

FRACTIONAL DIFFERENTIABILITY FOR SOLUTIONS OF NONLINEAR ELLIPTIC EQUATIONS

A. L. Baisón, A. Clop, R. Giova, J. Orobitg, A. Passarelli di Napoli

Abstract

We study nonlinear elliptic equations in divergence form

$$\operatorname{div} \mathcal{A}(x, Du) = \operatorname{div} G.$$

When \mathcal{A} has linear growth in Du , and assuming that $x \mapsto \mathcal{A}(x, \xi)$ enjoys $B_{\frac{n}{\alpha}, q}^\alpha$ smoothness, local well-posedness is found in $B_{p, q}^\alpha$ for certain values of $p \in [2, \frac{n}{\alpha})$ and $q \in [1, \infty]$. In the particular case $\mathcal{A}(x, \xi) = A(x)\xi$, $G = 0$ and $A \in B_{\frac{n}{\alpha}, q}^\alpha$, $1 \leq q \leq \infty$, we obtain $Du \in B_{p, q}^\alpha$ for each $p < \frac{n}{\alpha}$. Our main tool in the proof is a more general result, that holds also if \mathcal{A} has growth $s - 1$ in Du , $2 \leq s \leq n$, and asserts local well-posedness in L^q for each $q > s$, provided that $x \mapsto \mathcal{A}(x, \xi)$ satisfies a locally uniform *VMO* condition.

1 Introduction

The main purpose of this paper consists of analyzing the extra *fractional* differentiability of weak solutions of the following nonlinear elliptic equations in divergence form,

$$\operatorname{div} \mathcal{A}(x, Du) = \operatorname{div} G \quad \text{in } \Omega, \quad (1.1)$$

where $\Omega \subset \mathbb{R}^n$ is a domain, $u : \Omega \rightarrow \mathbb{R}$, $G : \Omega \rightarrow \mathbb{R}^n$, and $\mathcal{A} : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a *Carathéodory function with linear growth*. This means that there are constants $\ell, L, \nu > 0$ and $0 \leq \mu \leq 1$ such that

$$(A1) \quad \langle \mathcal{A}(x, \xi) - \mathcal{A}(x, \eta), \xi - \eta \rangle \geq |\xi - \eta|^2,$$

$$(A2) \quad |\mathcal{A}(x, \xi) - \mathcal{A}(x, \eta)| \leq L|\xi - \eta|,$$

$$(A3) \quad |\mathcal{A}(x, \xi)| \leq \ell(\mu^2 + |\xi|^2)^{\frac{1}{2}},$$

for every $\xi, \eta \in \mathbb{R}^n$ and for a.e. $x \in \Omega$.

It is clear that no extra differentiability can be expected for solutions, even if G is smooth, unless some assumption is given on the x -dependence of \mathcal{A} . Thus, we wish to find conditions on \mathcal{A} under which

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fractional differentiability assumptions on G transfer to Du with no losses in the order of differentiation.

The regularity theory for elliptic equations goes back to the seminal works by de Giorgi, Nash and Moser on Hölder continuity of weak solutions. Later on, for linear equations, Meyers found the existence of a number $p_0(n, \nu, L, \ell)$ such that a priori L^p estimates for the gradient hold whenever $p'_0 < p < p_0$. In both cases, no regularity for the coefficients is needed (other than measurability). Also, both the C^α and the L^p theory have been extended to nonlinear Carathéodory functions \mathcal{A} not necessarily having linear growth (we refer the interested reader to the monographs [6] and [8] for a complete treatment of the subject). If one seeks for higher differentiability results, then extra assumptions are needed on the coefficients. The classical Schauder estimates are a typical example of this fact, and can be used to prove that Hölder regularity on the independent term G transfers to the gradient Du in a nice way, provided the dependence $x \mapsto \mathcal{A}(x, \xi)$ is also Hölder. A very precise and unified description of such phenomenon can be found at Kuusi-Mingione [19].

Even though there is an extensive literature on the regularity theory for equations like (1.1), recent works in the planar situation, $n = 2$, have shown a renovated interest in determining the higher differentiability of solutions in terms of that of the datum and the coefficients. So far, especial attention has been driven to the case of fractional Sobolev spaces $W^{\alpha, p}$. It turns out that remarkable differences are appreciated, depending on the quantity αp :

- If $\alpha p > 2$, then $G \in W^{\alpha, q}$ implies $Du \in W^{\alpha, q}$ whenever $q \leq p$ (see e.g. references [5] and [18]).
- If $\alpha p = 2$, then $G \in W^{\alpha, q}$ implies $Du \in W^{\alpha, q}$ for every $q < p$, but not if $q = p$. The reason is that coefficients in $W^{\alpha, \frac{2}{\alpha}}_{loc}$ do not necessarily imply bounded derivative solutions. Precise results in this direction are given in [3] (for $\alpha = 1$) or [2] (for $0 < \alpha < 1$).
- If $\alpha p < 2$, then $G \in W^{\alpha, q}$ implies $Du \in W^{\alpha, q}$ for $q < q_0$ where q_0 depends on the ellipticity constants, and is such that $q_0 < p$. See for instance [3] for the case $\alpha = 1$, and [4] for $0 < \alpha < 1$.

The results mentioned above refer to the planar Beltrami equation, which is equivalent to $\mathcal{A}(x, \xi) = A(x)\xi$ for some $A(x)$ which is symmetric and has determinant 1.

It turns out that similar phenomena seem to occur in higher dimensions. Indeed, recent developments for nonlinear equations suggest that linearity should not be a restriction, as appropriate counterparts hold even if \mathcal{A} has superlinear growth, see for instance [7], [20] and [21]. In these works, higher differentiability is obtained from a pointwise condition on the partial map \mathcal{A} . More precisely, for Carathéodory functions \mathcal{A} with linear growth, it is assumed that there exists a non negative function $g \in L^n_{loc}(\Omega)$ such that

$$|\mathcal{A}(x, \xi) - \mathcal{A}(y, \xi)| \leq |x - y| (g(x) + g(y)) (\mu^2 + |\xi|^2)^{\frac{1}{2}}, \quad (1.2)$$

for almost every $x, y \in \Omega$, and every $\xi \in \mathbb{R}^n$. Under this condition, solutions to (1.1) with $G = 0$ are shown in [21] to be such that $Du \in W^{1, p}_{loc}$ for every $p < 2$. As a first fractional counterpart to this

result, instead of (1.2) one can assume that there is $g \in L_{loc}^{\frac{n}{\alpha}}(\Omega)$ such that

$$|\mathcal{A}(x, \xi) - \mathcal{A}(y, \xi)| \leq |x - y|^\alpha (g(x) + g(y)) (\mu^2 + |\xi|^2)^{\frac{1}{2}}, \quad (1.3)$$

for almost every $x, y \in \Omega$, and every $\xi \in \mathbb{R}^n$. It turns out that one gets improved regularity for solutions measured in terms of the Besov spaces $B_{p,q}^\alpha$.

Theorem 1. *Let $0 < \alpha < 1$. Assume that \mathcal{A} satisfies (A1), (A2), (A3), and that (1.3) holds for some $g \in L_{loc}^{\frac{n}{\alpha}}$. There exists $p_0 = p_0(n, \nu, \ell) > 2$ such that if $u \in W_{loc}^{1,2}$ is a weak solution of*

$$\operatorname{div} \mathcal{A}(x, Du) = 0$$

then $Du \in B_{p,\infty}^\alpha$, locally, whenever $2 \leq p < \min\{\frac{n}{\alpha}, p_0\}$. If $\mathcal{A}(x, \xi) = A(x)\xi$ then $2 \leq p < \frac{n}{\alpha}$ suffices.

See Section 2 for the definition of $B_{p,q}^\alpha$ and the meaning of *locally*. Theorem 1 holds even for some values $p > 2$, that is, different than the natural summability for \mathcal{A} . The reason for this is a non-standard version of the Caccioppoli inequality, see Lemma 14 in Section 2. At the same time, Theorem 1 seems to be in contrast with the results at [21]. Indeed, condition (1.2) fully describes equations with coefficients in the Sobolev space $W^{1,n}$, that is, the Triebel-Lizorkin space $F_{n,2}^1$ (see [17] for details), and so in [21] the Triebel-Lizorkin scale is nicely transferred from coefficients to solutions. Nevertheless, it is worth mentioning here that there is a jump between (1.2) and (1.3), since (1.3) says that $A \in F_{\frac{n}{\alpha},\infty}^\alpha$ (see [17] for details).

Unfortunately, the method does not seem to extend to the existing counterparts of (1.3) that characterize coefficients in $F_{\frac{n}{\alpha},q}^\alpha$ for finite values of q . Somewhat surprisingly, the Besov setting fits better in this context. To be precise, given $0 < \alpha < 1$ and $1 \leq q \leq \infty$ we say that (A4) is satisfied if there exists a sequence of measurable non-negative functions $g_k \in L^{\frac{n}{\alpha}}(\Omega)$ such that

$$\sum_k \|g_k\|_{L^{\frac{n}{\alpha}}(\Omega)}^q < \infty,$$

and at the same time

$$|\mathcal{A}(x, \xi) - \mathcal{A}(y, \xi)| \leq |x - y|^\alpha (g_k(x) + g_k(y)) (1 + |\xi|^2)^{\frac{1}{2}}$$

for each $\xi \in \mathbb{R}^n$ and almost every $x, y \in \Omega$ such that $2^{-k} \leq |x - y| < 2^{-k+1}$. We will shortly write then that $(g_k)_k \in \ell^q(L^{\frac{n}{\alpha}})$. If $\mathcal{A}(x, \xi) = A(x)\xi$ and $\Omega = \mathbb{R}^n$ then (A4) says that the entries of A belong to $B_{\frac{n}{\alpha},q}^\alpha$, see [17, Theorem 1.2]. Under (A4), we can prove the following result.

Theorem 2. *Let $0 < \alpha < 1$ and $1 \leq q \leq \infty$, and assume that \mathcal{A} satisfies (A1), (A2), (A3), (A4). There exists a number $p_0 = p_0(n, \nu, \ell) > 2$ such that if $u \in W_{loc}^{1,2}(\Omega)$ is a weak solution of*

$$\operatorname{div} \mathcal{A}(x, Du) = 0$$

then $Du \in B_{p,q}^\alpha$ provided that $2 \leq p < \min\{\frac{n}{\alpha}, p_0\}$. If $\mathcal{A}(x, \xi) = A(x)\xi$ then $2 \leq p < \frac{n}{\alpha}$ suffices.

Theorem 2 extends Theorem 1, because (1.3) implies (A4) (indeed $F_{\frac{n}{\alpha},\infty}^\alpha \subset B_{\frac{n}{\alpha},\infty}^\alpha$). The situation changes drastically if one looks at the inhomogeneous equation (1.1). Difficulties appear mainly with the third index q , due to the fact that if $1 \leq p < \frac{n}{\alpha}$ and $p_\alpha^* = \frac{np}{n-\alpha p}$ then the embedding $B_{p,q}^\alpha \subset L^{p_\alpha^*}$ only holds if $1 \leq q \leq p_\alpha^*$, and fails otherwise. We obtain the following result.

Theorem 3. *Let $0 < \alpha < 1$, and $1 \leq q \leq \infty$. Assume that \mathcal{A} satisfies $(\mathcal{A}1), (\mathcal{A}2), (\mathcal{A}3), (\mathcal{A}4)$. There exists a number $p_0 = p_0(n, \nu, \ell) > 2$ such that the implication*

$$G \in B_{p,q}^\alpha \quad \Rightarrow \quad Du \in B_{p,q}^\alpha$$

holds locally, provided that $\max\{p'_0, \frac{nq}{n+\alpha q}\} < p < \min\{\frac{n}{\alpha}, p_0\}$ and u, G satisfy (1.1).

The above theorem is sharp, in the sense that one cannot expect Du to belong to a Besov space $B_{p',q'}^\beta$ for any $\beta > \alpha$ (as can be seen from the constant coefficient setting). Moreover, our arguments also show that the restriction $p'_0 < p < p_0$ can be removed if \mathcal{A} is linear in the gradient variable. In fact, we have the following

Theorem 4. *Let $0 < \alpha < 1$ and $1 \leq q \leq \infty$, and assume that $\mathcal{A}(x, \xi) = A(x)\xi$ satisfies $(\mathcal{A}1), (\mathcal{A}2), (\mathcal{A}3)$. Suppose that the entries of $A(x)$ belong to $B_{\frac{n}{\alpha},q}^\alpha$. Then the implication*

$$G \in B_{p,q}^\alpha \quad \Rightarrow \quad Du \in B_{p,q}^\alpha$$

holds locally, provided that $\max\{1, \frac{nq}{n+\alpha q}\} < p < \frac{n}{\alpha}$ and u, G satisfy (1.1).

We do not know if Theorems 2, 3 and 4 remain true in the Triebel-Lizorkin setting, that is, replacing $\ell^q(L^{\frac{n}{\alpha}})$ by $L^{\frac{n}{\alpha}}(\ell^q)$ and $B_{p,q}^\alpha$ by $F_{p,q}^\alpha$.

Theorems 1, 2, 3 and 4 rely on the basic fact that the Besov spaces $B_{\frac{n}{\alpha},q}^\alpha$ and the Triebel-Lizorkin space $F_{\frac{n}{\alpha},\infty}^\alpha$ continuously embed into the VMO space of Sarason (e.g. [11, Prop. 7.12]). Linear equations with VMO coefficients are known to have a nice L^p theory (see [12] for $n = 2$ or [14] for $n \geq 2$). A first nonlinear growth counterpart was found in [15] for $\mathcal{A}(x, \xi) = \langle A(x)\xi, \xi \rangle^{s-2} A(x)\xi$, $2 \leq s \leq n$. Given $s \in [2, n]$, we say that $\mathbf{A} : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a *Carathéodory function of growth $s - 1$* if there are constants $L, \ell, \nu > 0$ and $0 \leq \mu \leq 1$ such that

$$(\mathbf{A}1) \quad \langle \mathbf{A}(x, \xi) - \mathbf{A}(x, \eta), \xi - \eta \rangle \geq \nu(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{s-2}{2}} |\xi - \eta|^2,$$

$$(\mathbf{A}2) \quad |\mathbf{A}(x, \xi) - \mathbf{A}(x, \eta)| \leq L(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{s-2}{2}} |\xi - \eta|, \text{ and}$$

$$(\mathbf{A}3) \quad |\mathbf{A}(x, \xi)| \leq \ell(\mu^2 + |\xi|^2)^{\frac{s-1}{2}},$$

The following result is our main tool for proving Theorems 1, 2, 3 and 4.

Theorem 5. *Let $2 \leq s \leq n$, and $q > s$. Assume that $(\mathbf{A}1), (\mathbf{A}2), (\mathbf{A}3)$ hold, and also that $x \mapsto \mathbf{A}(x, \xi)$ is locally uniformly in VMO . If u is a weak solution of*

$$\operatorname{div} \mathbf{A}(x, Du) = \operatorname{div}(|G|^{s-2} G) \tag{1.4}$$

with $G \in L_{loc}^q$, then $Du \in L_{loc}^q$. Moreover, there exists a constant $\lambda > 1$ such that the Caccioppoli inequality

$$\int_B |Du|^q \leq C(n, \lambda, \nu, \ell, L, s, q) \left(1 + \frac{1}{|B|^{q/n}} \int_{\lambda B} |u|^q + \int_{\lambda B} |G|^q \right)$$

holds for any ball B such that $\lambda B \subset \Omega$.

See Section 3 for the precise definition of *locally uniformly VMO*. The proof of Theorem 5 is inspired by that of [15], although now new technical difficulties arise from the fully nonlinear structure of \mathcal{A} . The result has its own interest, especially for two reasons. First, $\xi \mapsto \mathbf{A}(x, \xi)$ is not assumed to be \mathcal{C}^1 -smooth. Second, the allowed independent terms do not restrict to finite measures. Under these assumptions many interesting bounds on the size and the oscillations of the solutions and gradients are established in [18] and [19]. Unfortunately, and in contrast to the linear situation, this time the lack of self-adjointness is an obstacle to extend the result for values $q \in (1, s)$.

The paper is structured as follows. In Section 2 we give some preliminaries on Harmonic Analysis. In Section 3 we prove Theorem 5. In Section 4 we prove Theorem 1 as it is illustrative for proving Theorem 2 later. In Section 5 we prove Theorems 2, 3 and Theorem 4 .

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2 Notations and Preliminary Results

In this paper we follow the usual convention and denote by c a general constant that may vary on different occasions, even within the same line of estimates. Relevant dependencies on parameters and special constants will be suitably emphasized using parentheses or subscripts. The norm we use on \mathbb{R}^n will be the standard euclidean one and it will be denoted by $|\cdot|$. In particular, for vectors $\xi, \eta \in \mathbb{R}^n$ we write $\langle \xi, \eta \rangle$ for the usual inner product of ξ and η , and $|\xi| := \langle \xi, \xi \rangle^{\frac{1}{2}}$ for the corresponding euclidean norm. In what follows, $B(x, r) = B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$ will denote the ball centered at x of radius r . We shall omit the dependence on the center and on the radius when no confusion arises.

2.1 Besov-Lipschitz spaces

Given $h \in \mathbb{R}^n$ and $v : \mathbb{R}^n \rightarrow \mathbb{R}$, let us denote $\tau_h v(x) = v(x + h)$ and $\Delta_h v(x) = v(x + h) - v(x)$. As in [22, Section 2.5.12], given $0 < \alpha < 1$ and $1 \leq p, q < \infty$, we say that v belongs to the Besov space $B_{p,q}^\alpha(\mathbb{R}^n)$ if $v \in L^p(\mathbb{R}^n)$ and

$$\|v\|_{B_{p,q}^\alpha(\mathbb{R}^n)} = \|v\|_{L^p(\mathbb{R}^n)} + [v]_{\dot{B}_{p,q}^\alpha(\mathbb{R}^n)} < \infty$$

where

$$[v]_{\dot{B}_{p,q}^\alpha(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{|v(x+h) - v(x)|^p}{|h|^{\alpha p}} dx \right)^{\frac{q}{p}} \frac{dh}{|h|^n} \right)^{\frac{1}{q}} < \infty$$

Equivalently, we could simply say that $v \in L^p(\mathbb{R}^n)$ and $\frac{\Delta_h v}{|h|^\alpha} \in L^q\left(\frac{dh}{|h|^n}; L^p(\mathbb{R}^n)\right)$. As usually, if one simply integrates for $h \in B(0, \delta)$ for a fixed $\delta > 0$ then an equivalent norm is obtained, because

$$\left(\int_{\{|h| \geq \delta\}} \left(\int_{\mathbb{R}^n} \frac{|v(x+h) - v(x)|^p}{|h|^{\alpha p}} dx \right)^{\frac{q}{p}} \frac{dh}{|h|^n} \right)^{\frac{1}{q}} \leq c(n, \alpha, p, q, \delta) \|v\|_{L^p(\mathbb{R}^n)}$$

Similarly, we say that $v \in B_{p,\infty}^\alpha(\mathbb{R}^n)$ if $v \in L^p(\mathbb{R}^n)$ and

$$[v]_{\dot{B}_{p,\infty}^\alpha(\mathbb{R}^n)} = \sup_{h \in \mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{|v(x+h) - v(x)|^p}{|h|^{\alpha p}} dx \right)^{\frac{1}{p}} < \infty$$

Again, one can simply take supremum over $|h| \leq \delta$ and obtain an equivalent norm. By construction, $B_{p,q}^\alpha(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$. One also has the following version of Sobolev embeddings (a proof can be found at [11, Prop. 7.12]).

Lemma 6. *Suppose that $0 < \alpha < 1$.*

(a) *If $1 < p < \frac{n}{\alpha}$ and $1 \leq q \leq p_\alpha^*$ then there is a continuous embedding $B_{p,q}^\alpha(\mathbb{R}^n) \subset L^{p_\alpha^*}(\mathbb{R}^n)$.*

(b) *If $p = \frac{n}{\alpha}$ and $1 \leq q \leq \infty$ then there is a continuous embedding $B_{p,q}^\alpha(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$.*

Given a domain $\Omega \subset \mathbb{R}^n$, we say that v belongs to the local Besov space $B_{p,q,loc}^\alpha$ if φv belongs to the global Besov space $B_{p,q}^\alpha(\mathbb{R}^n)$ whenever φ belongs to the class $\mathcal{C}_c^\infty(\Omega)$ of smooth functions with compact support contained in Ω . The following Lemma is an easy exercise.

Lemma 7. *A function $v \in L_{loc}^p(\Omega)$ belongs to the local Besov space $B_{p,q,loc}^\alpha$ if and only if*

$$\left\| \frac{\Delta_h v}{|h|^\alpha} \right\|_{L^q\left(\frac{dh}{|h|^n}; L^p(B)\right)} < \infty$$

for any ball $B \subset 2B \subset \Omega$ with radius r_B . Here the measure $\frac{dh}{|h|^n}$ is restricted to the ball $B(0, r_B)$ on the h -space.

Proof. Let us fix a smooth and compactly supported test function φ . We have the pointwise identity

$$\frac{\Delta_h(\varphi v)(x)}{|h|^\alpha} = v(x+h) \frac{\Delta_h \varphi(x)}{|h|^\alpha} + \frac{\Delta_h v(x)}{|h|^\alpha} \varphi(x).$$

It is clear that

$$\left| v(x+h) \frac{\Delta_h \varphi(x)}{|h|^\alpha} \right| \leq |v(x+h)| \|\nabla \varphi\|_\infty |h|^{1-\alpha}$$

and therefore one always has $\frac{\Delta_h \varphi}{|h|^\alpha} \in L^q\left(\frac{dh}{|h|^n}; L^p(\mathbb{R}^n)\right)$. As a consequence, we have the equivalence

$$\varphi v \in B_{p,q}^\alpha(\mathbb{R}^n) \iff \frac{\Delta_h v}{|h|^\alpha} \varphi \in L^q\left(\frac{dh}{|h|^n}; L^p(\mathbb{R}^n)\right).$$

However, it is clear that $\frac{\Delta_h v}{|h|^\alpha} \varphi \in L^q\left(\frac{dh}{|h|^n}; L^p(\mathbb{R}^n)\right)$ for each $\varphi \in \mathcal{C}_c^\infty(\Omega)$ if and only if the same happens for every $\varphi = \chi_B$ and every ball $B \subset 2B \subset \Omega$. The claim follows. \square

As in [22, Section 2.5.10], we say that a function $v : \mathbb{R}^n \rightarrow \mathbb{R}$ belongs to the Triebel-Lizorkin space $F_{p,q}^\alpha(\mathbb{R}^n)$ if $v \in L^p(\mathbb{R}^n)$ and

$$\|v\|_{F_{p,q}^\alpha(\mathbb{R}^n)} = \|v\|_{L^p(\mathbb{R}^n)} + [v]_{\dot{F}_{p,q}^\alpha(\mathbb{R}^n)} < \infty,$$

where

$$[v]_{\dot{F}_{p,q}^\alpha(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{|v(x+h) - v(x)|^q}{|h|^{n+\alpha q}} dh \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}$$

Equivalently, we could simply say that $v \in L^p(\mathbb{R}^n)$ and $\frac{\Delta_h v}{|h|^\alpha} \in L^p\left(dx; L^q\left(\frac{dh}{|h|^n}\right)\right)$.

It turns out that Besov-Lipschitz and Triebel-Lizorkin spaces of fractional order $\alpha \in (0, 1)$ can be characterized in pointwise terms. Given a measurable function $v : \mathbb{R}^n \rightarrow \mathbb{R}$, a *fractional α -Hajlasz gradient for v* is a sequence $(g_k)_k$ of measurable, non-negative functions $g_k : \mathbb{R}^n \rightarrow \mathbb{R}$, together with a null set $N \subset \mathbb{R}^n$, such that the inequality

$$|v(x) - v(y)| \leq |x - y|^\alpha (g_k(x) + g_k(y))$$

holds whenever $k \in \mathbb{Z}$ and $x, y \in \mathbb{R}^n \setminus N$ are such that $2^{-k} \leq |x - y| < 2^{-k+1}$. We say that $(g_k) \in \ell^q(\mathbb{Z}; L^p(\mathbb{R}^n))$ if

$$\|(g_k)_k\|_{\ell^q(L^p)} = \left(\sum_{k \in \mathbb{Z}} \|g_k\|_{L^p(\mathbb{R}^n)}^q \right)^{\frac{1}{q}} < \infty.$$

Similarly, we write $(g_k) \in L^p(\mathbb{R}^n; \ell^q(\mathbb{Z}))$ if

$$\|(g_k)_k\|_{L^p(\ell^q)} = \left(\int_{\mathbb{R}^n} \|(g_k(x))_k\|_{\ell^q(\mathbb{Z})}^p dx \right)^{\frac{1}{p}} < \infty.$$

The following result was proven in [17].

Theorem 8. *Let $0 < \alpha < 1$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Let $v \in L^p(\mathbb{R}^n)$.*

1. *One has $v \in B_{p,q}^\alpha(\mathbb{R}^n)$ if and only if there exists a fractional α -Hajlasz gradient $(g_k)_k \in \ell^q(\mathbb{Z}; L^p(\mathbb{R}^n))$ for v . Moreover,*

$$\|v\|_{B_{p,q}^\alpha(\mathbb{R}^n)} \simeq \inf \|(g_k)_k\|_{\ell^q(L^p)}$$

where the infimum runs over all possible fractional α -Hajlasz gradients for v .

2. *One has $v \in F_{p,q}^\alpha(\mathbb{R}^n)$ if and only if there exists a fractional α -Hajlasz gradient $(g_k)_k \in L^p(\mathbb{R}^n; \ell^q(\mathbb{Z}))$ for v . Moreover,*

$$\|v\|_{F_{p,q}^\alpha(\mathbb{R}^n)} \simeq \inf \|(g_k)_k\|_{L^p(\ell^q)}$$

where the infimum runs over all possible fractional α -Hajlasz gradients for v .

2.2 Nonlinear elliptic equations in divergence form

This section is devoted to recall some fundamental results of L_{loc}^p -theory for solutions of nonlinear elliptic equations in divergence form that we shall use in what follows. Our first result is a very well known higher integrability property, that we state in a version suitable for our purposes.

Theorem 9. Let $s \in [2, n]$, and let $\mathbf{A} : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfy (A1)–(A3). Let $u \in W_{\text{loc}}^{1,s}(\Omega)$ be a local solution of (1.4). If $G \in L_{\text{loc}}^q(\Omega)$ for some $q > s$, then there exists an exponent t , $s < t < q$ such that $Du \in L_{\text{loc}}^t(\Omega)$. Moreover, the following estimate

$$\left(\int_{B_R} |Du|^t dx \right)^{\frac{1}{t}} \leq C \left(\int_{B_{2R}} |Du|^s dx \right)^{\frac{1}{s}} + \left(\int_{B_{2R}} |G|^t dx \right)^{\frac{1}{t}},$$

holds for every ball $B_R \subset B_{2R} \Subset \Omega$.

For the proof we refer to [8, Theorem 6.7, p. 204]. Next, we state a regularity result for solutions of homogenous non linear elliptic equations of the form

$$\operatorname{div} \mathcal{B}(Du) = 0$$

where $\mathcal{B} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ an autonomous Carathéodory function with growth $s - 1$. This means that

$$(B1) \quad \langle \mathcal{B}(\xi) - \mathcal{B}(\eta), \xi - \eta \rangle \geq \nu(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{s-2}{2}} |\xi - \eta|^2,$$

$$(B2) \quad |\mathcal{B}(\xi) - \mathcal{B}(\eta)| \leq L(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{s-2}{2}} |\xi - \eta|, \text{ and}$$

$$(B3) \quad |\mathcal{B}(\xi)| \leq \ell(\mu^2 + |\xi|^2)^{\frac{s-1}{2}},$$

for each $\xi, \eta \in \mathbb{R}^n$. We recall the following.

Theorem 10. Let $\mathcal{B} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be such that (B1), (B2), (B3) hold, and $v \in W_{\text{loc}}^{1,s}(\Omega)$ be a solution of

$$\operatorname{div} \mathcal{B}(Dv) = 0 \quad \text{in } \Omega,$$

Then, for every ball $B \Subset \Omega$, we have

- $\sup_{x \in \lambda B} |Dv(x)| \leq \frac{C}{\operatorname{diam}(B)(1-\lambda)} (f_B(1 + |Dv|^s))^{\frac{1}{s}}$ for all $0 < \lambda < 1$.
- $f_{\delta B} |Dv - (Dv)_{\delta B}|^s \leq C \delta^{\alpha s} f_B(1 + |Dv|^s)$ for all $0 < \delta < 1$ and some $\alpha > 0$.

For the proof we refer to Sections 8.3 and 8.7 in [8] or, more specifically to formulas (8.104) and (8.106), p.302-303 in [8]. From previous Theorem, one can easily deduce the following.

Lemma 11. Let $\mathcal{B} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be such that (B1), (B2), (B3) hold. Let $B \Subset \Omega$ be a ball, and let $w \in W_{\text{loc}}^{1,s}(\Omega)$. Then the problem

$$\begin{cases} \operatorname{div} \mathcal{B}(Dv) = 0 & x \in B, \\ v = w & x \in \partial B. \end{cases}$$

has a unique solution $v \in W^{1,s}(B)$. Moreover, one has:

- $\sup_{x \in \lambda B} |Dv(x)| \leq \frac{C}{\operatorname{diam}(B)(1-\lambda)} (f_B(1 + |Dw|^s))^{\frac{1}{s}}$ for all $0 < \lambda < 1$.
- $f_{\delta B} |Dv - (Dv)_{\lambda B}|^s \leq C \delta^{\alpha s} f_B(1 + |Dw|^s)$ for all $0 < \delta < 1$ and some $\alpha > 0$.

We conclude this section with a very well known iteration Lemma, that finds an important application in the so called hole-filling method. Its proof can be found for example in [8, Lemma 6.1].

Lemma 12. *Let $h : [r, R_0] \rightarrow \mathbb{R}$ be a non-negative bounded function and $0 < \vartheta < 1$, $A, B \geq 0$ and $\beta > 0$. Assume that*

$$h(s) \leq \vartheta h(t) + \frac{A}{(t-s)^\beta} + B,$$

for all $r \leq s < t \leq R_0$. Then

$$h(r) \leq \frac{cA}{(R_0-r)^\beta} + cB,$$

where $c = c(\vartheta, \beta) > 0$.

2.3 Hodge decomposition

The interested reader can check the contents of this section in the monograph [13]. We recall that for a vector field $F \in L^p(\mathbb{R}^n, \mathbb{R}^n)$, with $1 < p < +\infty$, the Poisson equation

$$\Delta w = \operatorname{div} F$$

is solved by a function $w \in W^{1,p}$ whose gradient can be expressed in terms of the Riesz transform as follows

$$Dw = -(\mathbf{R} \otimes \mathbf{R})(F).$$

The tensor product operator $\mathbf{R} \otimes \mathbf{R}$ is the $n \times n$ matrix whose entries are the second order Riesz transforms $R_j \circ R_k$ ($1 \leq j, k \leq n$) and therefore the above identity reads as

$$D_j w = - \sum_{k=1}^n R_j R_k F^k,$$

where F^k denotes the k -th component of the vector field F . Setting $\mathbf{E} = -(\mathbf{R} \otimes \mathbf{R})$ and $\mathbf{B} = \mathbf{Id} - \mathbf{E}$ we then have that

$$F = \mathbf{E}(F) + \mathbf{B}(F).$$

By construction, $\mathbf{E}(F)$ is curl free and $\mathbf{B}(F)$ is divergence free. Standard Calderon-Zygmund theory yields L^p bounds for the operators \mathbf{E} and \mathbf{B} , whenever $1 < p < +\infty$. However, we will need a more precise estimate, which is contained in the following stability property of the Hodge decomposition.

Lemma 13. *Let $w \in W^{1,p}(\mathbb{R}^n)$, and let $1 < p < \infty$. Then there exist vector fields $\mathbf{E} \in L^{p'}(\mathbb{R}^n)$ with $\operatorname{curl}(\mathbf{E}) = 0$ and $\mathbf{B} \in L^{p'}(\mathbb{R}^n)$ with $\operatorname{div}(\mathbf{B}) = 0$ such that*

$$Dw |Dw|^{p-2} = \mathbf{E} + \mathbf{B}. \tag{2.1}$$

Moreover

$$\|\mathbf{E}\|_{L^{p'}(\mathbb{R}^n)} \leq C \|Dw\|_{L^p(\mathbb{R}^n)}^{p-1} \tag{2.2}$$

and

$$\|\mathbf{B}\|_{L^{p'}(\mathbb{R}^n)} \leq C \max\{p-2, p'-2\} \|Dw\|_{L^p(\mathbb{R}^n)}^{p-1}, \tag{2.3}$$

where C is a universal constant.

The proof of previous Lemma is contained in [14, Theorem 4]. The fact that the constant is independent of n and p can be derived as in [16, Corollary 3]. We use the above Hodge decomposition to prove the following non-standard Caccioppoli inequality, which is well-known for the community and whose proof is included for the reader's convenience.

Lemma 14. *Let \mathcal{A} be such that $(\mathcal{A}1), (\mathcal{A}3)$ hold. There exists a number $p_0 = p_0(n, \nu, L) > 2$ with the following property. If $p \in (p'_0, p_0)$ and $u \in W_{loc}^{1,p}(\Omega)$ is a weak solution of (1.1) for some $G \in L_{loc}^2 \cap L_{loc}^p$ then*

$$\int_{B_0} |Du|^p \leq C \left(\mu^p + \frac{1}{|B_0|^{p/n}} \int_{2B_0} |u|^p + \int_{2B_0} |G|^p \right)$$

for every ball $B_0 \subset 2B_0 \subset \Omega$.

Proof. Let B_r be a ball of radius r , such that $B_r \subset 2B_r \subset \Omega$. Choose radii $r < s < t < 2r$, and let $\eta \in C_c^\infty(\Omega)$ be a cut off function such that $\chi_{B_s} \leq \eta \leq \chi_{B_t}$, and $\|\nabla \eta\|_\infty \leq \frac{c}{t-s}$. We apply Lemma 13 to $w = \eta u$, so that we can write

$$|Dw|^{p-2} Dw = \mathbf{E} + \mathbf{B}$$

with $\mathbf{E}, \mathbf{B} \in L^{p'}(\mathbb{R}^n)$, both supported on B_t , $\text{div}(\mathbf{B}) = 0$, $\text{curl}(\mathbf{E}) = 0$, and moreover

$$\begin{aligned} \|\mathbf{E}\|_{L^{p'}(B_t)} &\leq C \|Dw\|_{L^p(B_t)}^{p-1}, \\ \|\mathbf{B}\|_{L^{p'}(B_t)} &\leq C \max\{p-2, p'-2\} \|Dw\|_{L^p(B_t)}^{p-1}. \end{aligned} \quad (2.4)$$

From $\text{curl}(\mathbf{E}) = 0$ and $1 < p' < \infty$ we know that there is $\varphi \in W_0^{1,p'}(B_t)$ such that $\mathbf{E} = D\varphi$. Now we test (1.1) with φ , and obtain

$$\int_{B_t} \langle \mathcal{A}(x, Du), Dw \rangle |Dw|^{p-2} = \int_{B_t} \langle \mathcal{A}(x, Du), \mathbf{B} \rangle + \int_{B_t} \langle G, D\varphi \rangle$$

whence

$$\int_{B_t} \langle \mathcal{A}(x, Du), Du \rangle \eta |Dw|^{p-2} = - \int_{B_t} \langle \mathcal{A}(x, Du), D\eta \rangle u |Dw|^{p-2} + \int_{B_t} \langle \mathcal{A}(x, Du), \mathbf{B} \rangle + \int_{B_t} \langle G, D\varphi \rangle.$$

Using now $(\mathcal{A}1), (\mathcal{A}3)$ and the properties of η , we get

$$\nu \int_{B_s} |Du|^p \leq \ell \int_{B_t \setminus B_s} (\mu + |Du|) |D\eta| |u| |Dw|^{p-2} + \ell \int_{B_t} (\mu + |Du|) |\mathbf{B}| + \int_{B_t} |G| |D\varphi|.$$

Now, since $w = \eta u$, Young's inequality tells us that

$$\int_{B_t \setminus B_s} (\mu + |Du|) |D\eta| |u| |Dw|^{p-2} \leq C(p) \int_{B_t \setminus B_s} |u|^p |D\eta|^p + C(p) \int_{B_t \setminus B_s} |Du|^p + C(p) \mu^p |2B_r|$$

Also, by estimate (2.3) and Young's inequality,

$$\begin{aligned} \int_{B_t} (\mu + |Du|) |\mathbf{B}| &\leq \|\mu + |Du|\|_{L^p(B_t)} \|\mathbf{B}\|_{L^{p'}(B_t)} \\ &\leq C \max\{p-2, p'-2\} \|\mu + |Du|\|_{L^p(B_t)} \|Dw\|_{L^p(B_t)}^{p-1} \\ &\leq C 2^{p-2} \max\{p-2, p'-2\} \|\mu + |Du|\|_{L^p(B_t)} (\|u D\eta\|_{L^p(B_t)}^{p-1} + \|\eta Du\|_{L^p(B_t)}^{p-1}) \\ &\leq C(p) \mu^p |2B_r| + C 2^{p-2} \max\{p-2, p'-2\} \|Du\|_{L^p(B_t)}^p + C(p) \|u D\eta\|_{L^p(B_t)}^p \end{aligned}$$

where C is the universal constant from Lemma 13. Finally, also by (2.2) and Young's inequality we get

$$\begin{aligned}
\int_{B_t} |G| |D\varphi| &\leq \|G\|_{L^p(B_t)} \|D\varphi\|_{L^{p'}(B_t)} \\
&\leq \varepsilon \|D\varphi\|_{L^{p'}(B_t)}^{p'} + C(\varepsilon, p) \|G\|_{L^p(B_t)}^p \\
&\leq C\varepsilon \|Dw\|_{L^p(B_t)}^p + C(\varepsilon, p) \|G\|_{L^p(B_t)}^p \\
&\leq C 2^{p-1} \varepsilon \|Du\|_{L^p(B_t)}^p + C 2^{p-1} \varepsilon \|u D\eta\|_{L^p(B_t)}^p + C(\varepsilon, p) \|G\|_{L^p(B_t)}^p,
\end{aligned}$$

where $\varepsilon > 0$ will be chosen later. Putting this together,

$$\begin{aligned}
\nu \int_{B_s} |Du|^p &\leq C(p, \ell, \varepsilon) \int_{B_t} |u|^p |D\eta|^p + C(p, \ell) \int_{B_t \setminus B_s} |Du|^p + C(p) \mu^p |2B_r| \\
&\quad + C(\ell) 2^{p-2} (\max\{p-2, p'-2\} + 2\varepsilon) \|Du\|_{L^p(B_t)}^p + C(\varepsilon) \|G\|_{L^p(B_t)}^p.
\end{aligned}$$

Adding $C(p, \ell) \int_{B_s} |Du|^p$ at both sides, and using the properties of η , we get

$$\begin{aligned}
(\nu + C(p, \ell)) \int_{B_s} |Du|^p &\leq \frac{C(p, \ell, \varepsilon)}{(t-s)^p} \int_{2B_r} |u|^p + C(p) \mu^p |2B_r| \\
&\quad + \left(C(p, \ell) + C(\ell) 2^{p-2} (\max\{p-2, p'-2\} + 2\varepsilon) \right) \|Du\|_{L^p(B_t)}^p + C(\varepsilon) \|G\|_{L^p(B_t)}^p.
\end{aligned}$$

Above, it is clear that one can always attain

$$C(\ell) 2^{p-2} (\max\{p-2, p'-2\} + \varepsilon) \leq \frac{\nu}{2},$$

if $\varepsilon > 0$ is chosen small enough, and if p is chosen close enough to 2. We write this as $p \in (p'_0, p_0)$. At this point we can use the iteration Lemma 12 to finish the proof. \square

The number p_0 was precisely described in [1] when $n = s = 2$, and is unknown otherwise.

2.4 Maximal functions

Let $1 \leq s < \infty$, and let $u \in L_{loc}^s(\mathbb{R}^n; \mathbb{R})$. We will denote

$$\mathcal{M}_s(u)(x) = \sup_{r>0} \left(\int_{B(x,r)} |u|^s \right)^{\frac{1}{s}},$$

When $s = 1$ this is the classical Hardy-Littlewood maximal operator. We will also denote

$$\begin{aligned}
\mathcal{M}_s^\sharp(u)(x) &= \sup_{r>0} \left(\int_{B(x,r)} |u - u_{B(x,r)}|^s \right)^{\frac{1}{s}} \\
\mathcal{M}_{s,R}^\sharp(u)(x) &= \sup_{0<r<R} \left(\int_{B(x,r)} |u - u_{B(x,r)}|^s \right)^{\frac{1}{s}}
\end{aligned}$$

When $s = 1$ they go back to the well-known Fefferman-Stein sharp maximal function. These operators are classical tools in harmonic analysis, we refer the interested reader to [9, 10].

The following lemma is proven in [15] for $s = 1$. Its proof for $s > 1$ follows similarly.

Lemma 15. *Let $1 \leq s < q < \infty$, and let $u \in L^s(\mathbb{R}^n)$.*

(i) *One has $\|\mathcal{M}_s u\|_{L^q(\mathbb{R}^n)} \leq C(n, s, q)\|u\|_{L^q(\mathbb{R}^n)}$.*

(ii) *There exists a constant $k_0 = k_0(n, s, q) \geq 2$ such that if u is supported on a ball $B(x_0, R)$ then*

$$\|\mathcal{M}_{s, k_0 R}^\# u\|_{L^q(B(x_0, k_0 R))} \geq C(n, s, q)\|u\|_{L^q(B(x_0, R))}.$$

3 VMO coefficients in \mathbb{R}^n

In this section, we assume that $n \geq 2$, and that $\mathbf{A} : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a Caratheodory function such that assumptions (A1), (A2), (A3) in the introduction are satisfied. We also require a control on the oscillations, which is described as follows. Given a ball $B \subset \Omega$, let us denote

$$\mathbf{A}_B(\xi) = \oint_B \mathbf{A}(x, \xi) dx$$

One can easily check that the operator $\mathbf{A}_B(\xi)$ also satisfies assumption (A1), (A2), (A3). Now set

$$V(x, B) = \sup_{\xi \neq 0} \frac{|\mathbf{A}(x, \xi) - \mathbf{A}_B(\xi)|}{(\mu^2 + |\xi|^2)^{\frac{s-1}{2}}}, \quad (3.1)$$

for $x \in \Omega$ and $B \subset \Omega$. If \mathbf{A} is given by the weighted s -laplacian, that is $\mathbf{A}(x, \xi) = \gamma(x)|\xi|^{s-2}\xi$, one obtains

$$V(x, B) = |\gamma(x) - \gamma_B|,$$

where $\gamma_B = \int_B \gamma(y) dy$, and so any reasonable VMO condition on γ requires that the mean value of $V(x, B)$ on B goes to 0 as $|B| \rightarrow 0$. Our VMO assumption on general Carathéodory functions \mathbf{A} consists of a uniform version of this fact. Namely, we will say that $x \mapsto \mathbf{A}(x, \xi)$ is *locally uniformly in VMO* if for each compact set $K \subset \Omega$ we have that

$$\lim_{R \rightarrow 0} \sup_{r(B) < R} \sup_{c(B) \in K} \oint_B V(x, B) dx = 0. \quad (3.2)$$

Here $c(B)$ denotes the center of the ball B , and $r(B)$ its radius.

The main result in this section is an a priori estimate for weak solutions belonging to $W^{1,q}$ for some $q > s$. It is a local nonlinear version of the classical Theorem by Iwaniec and Sbordone [14]. Our proof relies on arguments similar to those used in [15].

Theorem 16. *Assume that \mathbf{A} satisfies (A1), (A2), (A3) and that it is locally uniformly in VMO, and let $q > s$. There exists $\lambda = \lambda(n, s, q) > 1$ with the following property. If $x_0 \in \Omega$ then there is a number $d_0 > 0$ (depending on $\nu, \ell, L, s, q, n, \mathbf{A}$ and x_0) such that if $u \in W^{1,q}(\Omega; \mathbb{R})$ is such that*

$$\operatorname{div} \mathbf{A}(x, Du) = \operatorname{div}(|G|^{s-2}G) \quad \text{weakly in } \Omega \quad (3.3)$$

for some $G \in L_{loc}^q(\Omega; \mathbb{R})$, then the estimate

$$\oint_{\mathbf{B}_0} |Du|^q \leq C \left(\mu^q + \frac{1}{d^q} \oint_{\lambda \mathbf{B}_0} |u|^q + \oint_{\lambda \mathbf{B}_0} |G|^q \right) \quad (3.4)$$

holds whenever $0 < d < d_0$, $\mathbf{B}_0 = B(x_0, d)$ and $\lambda \mathbf{B}_0 \subset \Omega$.

Proof of Theorem 16. Let $k_0 \geq 2$ be the constant in Lemma 15. Let $\delta \in (0, 1)$ to be determined later. We are given a ball $\mathbf{B}_0 = B(x_0, d)$, such that $\tilde{\mathbf{B}}_0 = (1 + \frac{2}{\delta})k_0\mathbf{B}_0 \subset \Omega$.

The first step consists of localizing the problem. This is done by choosing arbitrary radii $0 < \frac{d}{2} < \rho < r < d$, balls $\mathbf{B}_\rho = B(x_0, \rho)$ and $\mathbf{B}_r = B(x_0, r)$, and a cut off function $\eta : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\eta \in C_c^\infty(\mathbb{R}^n)$, $\chi_{\mathbf{B}_\rho} \leq \eta \leq \chi_{\mathbf{B}_r}$ and $\|D\eta\|_\infty \leq \frac{c(n)}{r-\rho}$. Set $w = \eta^{s'} u$. Then clearly $w \in W^{1,q}(\mathbb{R}^n)$ has compact support in \mathbf{B}_r and we have that

$$\begin{aligned} \operatorname{div} \mathbf{A}(x, Dw) &= \operatorname{div} (\mathbf{A}(x, Dw) - \mathbf{A}(x, \eta^{s'} Du)) + \operatorname{div} (\mathbf{A}(x, \eta^{s'} Du) - \eta^s \mathbf{A}(x, Du)) \\ &\quad + \operatorname{div} (\eta^s \mathbf{A}(x, Du)) \\ &= \operatorname{div} (\mathbf{A}(x, Dw) - \mathbf{A}(x, \eta^{s'} Du)) + \operatorname{div} (\mathbf{A}(x, \eta^{s'} Du) - \eta^s \mathbf{A}(x, Du)) \\ &\quad + \eta^s \operatorname{div} (\mathbf{A}(x, Du)) + D(\eta^s) \mathbf{A}(x, Du). \end{aligned} \tag{3.5}$$

For each $y \in k_0\mathbf{B}_0$ and each $0 < R < \frac{k_0 d}{\delta}$ we set $B_R = B(y, R)$. Then $B_R \subset (1 + \frac{1}{\delta})k_0\mathbf{B}_0 \subset \Omega$, and thus the quantity

$$\mathbf{A}_{B_R}(\xi) = \int_{B_R} \mathbf{A}(x, \xi) dx$$

is well defined. Let v be the unique solution to the following Dirichlet problem

$$\begin{cases} \operatorname{div} \mathbf{A}_{B_R}(Dv) = 0 & x \in B_R \\ v = w & x \in \partial B_R. \end{cases} \tag{3.6}$$

Now, we multiply both sides of the equality (3.5) by $v - w$ and, since $v - w$ vanishes outside of B_R , we can integrate by parts thus getting

$$\begin{aligned} \int_{B_R} \langle \mathbf{A}(x, Dw), Dv - Dw \rangle &= \int_{B_R} \langle \mathbf{A}(x, Dw) - \mathbf{A}(x, \eta^{s'} Du), Dv - Dw \rangle \\ &\quad + \int_{B_R} \langle \mathbf{A}(x, \eta^{s'} Du) - \eta^s \mathbf{A}(x, Du), Dv - Dw \rangle \\ &\quad + \int_{B_R} \langle \mathbf{A}(x, Du), D(\eta^s(v - w)) \rangle - \int_{B_R} D(\eta^s) \mathbf{A}(x, Du)(v - w) \\ &= \int_{B_R} \langle \mathbf{A}(x, Dw) - \mathbf{A}(x, \eta^{s'} Du), Dv - Dw \rangle \\ &\quad + \int_{B_R} \langle \mathbf{A}(x, \eta^{s'} Du) - \eta^s \mathbf{A}(x, Du), Dv - Dw \rangle \\ &\quad + \int_{B_R} |G|^{s-2} \langle G, D(\eta^s(v - w)) \rangle - \int_{B_R} D(\eta^s) \mathbf{A}(x, Du)(v - w), \end{aligned}$$

where, in the last equality, we used that u is a solution of the equation (3.3). On the other hand, since v is a solution of the Dirichlet problem (3.6), we also have

$$\int_{B_R} \langle \mathbf{A}_B(Dv) - \mathbf{A}_B(Dw), Dv - Dw \rangle = \int_{B_R} \langle \mathbf{A}_B(Dw), Dw - Dv \rangle$$

and so

$$\begin{aligned}
\int_{B_R} \langle \mathbf{A}_B(Dv) - \mathbf{A}_B(Dw), Dw - Dv \rangle &= \int_{B_R} \langle \mathbf{A}_B(Dw) - \mathbf{A}(x, Dw), Dw - Dv \rangle \\
&+ \int_{B_R} \langle \mathbf{A}(x, Dw) - \mathbf{A}(x, \eta^{s'} Du), Dw - Dv \rangle \\
&+ \int_{B_R} \langle \mathbf{A}(x, \eta^{s'} Du) - \eta^s \mathbf{A}(x, Du), Dw - Dv \rangle \\
&+ \int_{B_R} |G|^{s-2} \langle G, D(\eta^s(w-v)) \rangle - \int_{B_R} D(\eta^s) \mathbf{A}(x, Du)(w-v) \\
&\leq \int_{B_R} |\mathbf{A}_B(Dw) - \mathbf{A}(x, Dw)| |Dw - Dv| \\
&+ \int_{B_R} |\mathbf{A}(x, Dw) - \mathbf{A}(x, \eta^{s'} Du)| |Dw - Dv| \\
&+ \int_{B_R} |\mathbf{A}(x, \eta^{s'} Du) - \eta^s \mathbf{A}(x, Du)| |Dw - Dv| \\
&+ \int_{B_R} |G|^{s-1} |D(\eta^s(w-v))| + \int_{B_R} |D(\eta^s)| |\mathbf{A}(x, Du)| |w-v|.
\end{aligned}$$

We write previous inequality as follows

$$I_0 \leq I_1 + I_2 + I_3 + I_4 + I_5$$

and we estimate I_j separately. Since $s \geq 2$, by virtue of the ellipticity assumption (A1), we have that

$$\nu \int_{B_R} |Dv - Dw|^s \leq \nu \int_{B_R} |Dv - Dw|^2 (\mu^2 + |Dv|^2 + |Dw|^2)^{\frac{s-2}{2}} \leq I_0 \quad (3.7)$$

By the definition of $V(x, B)$ in (3.1), thanks to the assumption (3.2) and Young's and Hölder's inequalities, we estimate I_1 as follows

$$\begin{aligned}
I_1 &\leq \int_{B_R} V(x, B) (\mu^2 + |Dw|^2)^{\frac{s-1}{2}} |Dw - Dv| \\
&\leq \varepsilon \int_{B_R} |Dv - Dw|^s + C(\varepsilon, s) \int_B V(x, B)^{s'} (\mu^2 + |Dw|^2)^{\frac{s}{2}} \\
&\leq \varepsilon \int_{B_R} |Dv - Dw|^s + C(\varepsilon, s) \left(\int_{B_R} V(x, B)^{\frac{ts'}{t-s}} \right)^{\frac{t-s}{t}} \left(\int_{B_R} (\mu^2 + |Dw|^2)^{\frac{t}{2}} \right)^{\frac{s}{t}} \\
&\leq \varepsilon \int_{B_R} |Dv - Dw|^s + C(\varepsilon, s, t, \ell) \left(\int_{B_R} V(x, B) \right)^{\frac{t-s}{t}} \left(\int_{B_R} (\mu^2 + |Dw|^2)^{\frac{t}{2}} \right)^{\frac{s}{t}}, \quad (3.8)
\end{aligned}$$

where $t > s$ is the exponent determined in Lemma 9, $\varepsilon > 0$ is a parameter that will be chosen later and we used that the function $V(x, B)$, by virtue of assumption (A3), is bounded in Ω . By assumption (A2), the definition of w , Young's inequality and the properties of η , we have

$$I_2 \leq L \int_{B_R} |Dw - \eta^{s'} Du| (\mu^2 + |Dw|^2 + |\eta^{s'} Du|^2)^{\frac{s-2}{2}} |Dv - Dw|$$

$$\begin{aligned}
&= L \int_{B_R} |D(\eta^{s'})u|(\mu^2 + |Dw|^2 + |Dw - D(\eta^{s'})u|^2)^{\frac{s-2}{2}} |Dv - Dw| \\
&\leq c(s, L) \int_{B_R} |D(\eta^{s'})u|(\mu + |Dw| + |D(\eta^{s'})u|)^{s-2} |Dv - Dw| \\
&\leq c(s, L) \int_{B_R} |D(\eta^{s'})u|^{s-1} |Dv - Dw| + c(s) \int_{B_R} |D(\eta^{s'})u|(\mu + |Dw|)^{s-2} |Dv - Dw| \\
&\leq \varepsilon \int_{B_R} |Dv - Dw|^s + \sigma \int_{B_R} (\mu + |Dw|)^s + \frac{c(\varepsilon, \sigma, s)}{(r - \rho)^s} \int_{B_R} |u|^s, \tag{3.9}
\end{aligned}$$

where $\varepsilon, \sigma > 0$ will be chosen later. We now proceed with the estimate of I_3 . The properties of η and Young's inequality yield

$$\begin{aligned}
I_3 &\leq \int_{B_R \setminus \mathbf{B}_r} |\mathbf{A}(x, 0)| |Dv - Dw| + \int_{B_R \cap (\mathbf{B}_r \setminus \mathbf{B}_\rho)} |\mathbf{A}(x, \eta^{s'} Du) - \eta^s \mathbf{A}(x, Du)| |Dv - Dw| \\
&\leq C(\varepsilon) \int_{B_R \setminus \mathbf{B}_r} |\mathbf{A}(x, 0)|^{s'} + C(\varepsilon) \int_{B_R \cap (\mathbf{B}_r \setminus \mathbf{B}_\rho)} |\mathbf{A}(x, \eta^{s'} Du) - \mathbf{A}(x, Du)|^{s'} \\
&\quad + C(\varepsilon) \int_{B_R \cap (\mathbf{B}_r \setminus \mathbf{B}_\rho)} |\mathbf{A}(x, Du) - \eta^s \mathbf{A}(x, Du)|^{s'} + \varepsilon \int_{B_R} |Dv - Dw|^s \\
&\leq C(\varepsilon, \ell, s) \mu^s R^n + C(\varepsilon, L, s) \int_{B_R \cap (\mathbf{B}_r \setminus \mathbf{B}_\rho)} |\eta^{s'} Du - Du|^{s'} (\mu + |\eta^{s'} Du| + |Du|)^{s'(s-2)} \\
&\quad + C(\varepsilon, \ell, s) \int_{B_R \cap (\mathbf{B}_r \setminus \mathbf{B}_\rho)} (1 - \eta^s)^{s'} |Du|^s + \varepsilon \int_{B_R} |Dv - Dw|^s \\
&\leq C(\varepsilon, \ell, s) \mu^s R^n + C(\varepsilon, L, \ell, s) \int_{B_R} |Du|^s \chi_{\mathbf{B}_r \setminus \mathbf{B}_\rho} + \varepsilon \int_{B_R} |Dv - Dw|^s, \tag{3.10}
\end{aligned}$$

where we also used assumptions **(A2)** and **(A3)**. Using Young's inequality again and the properties of η , we have that

$$\begin{aligned}
I_4 &\leq \int_{B_R} \eta^s |G|^{s-1} |Dv - Dw| + \int_{B_R} |D(\eta^s)| |G|^{s-1} |v - w| \\
&\leq \varepsilon \int_{B_R} |Dv - Dw|^s + c(\varepsilon) \left(\frac{R^s}{(r - \rho)^s} + 1 \right) \int_{B_R} |G|^s + \varepsilon \int_{B_R} \frac{|v - w|^s}{R^s} \\
&\leq \varepsilon \int_{B_R} |Dv - Dw|^s + c(\varepsilon) \left(\frac{d^s k_0^s}{\delta^s (r - \rho)^s} + 1 \right) \int_{B_R} |G|^s + C(n, s) \varepsilon \int_{B_R} |Dv - Dw|^s, \tag{3.11}
\end{aligned}$$

where, in the last estimate, we used Poincaré - Wirtinger inequality and the bound $R < \frac{k_0 d}{\delta}$. Finally, by virtue of Young and Poincaré - Wirtinger inequalities and again the properties of η , we estimate

$$\begin{aligned}
I_5 &\leq \ell \int_{B_R} |D(\eta^s)| (\mu + |Du|)^{s-2} (\mu + |Du|) |v - w| \\
&\leq \frac{c(n, \ell)}{(r - \rho)} \int_{B_R} \eta^{\frac{1}{s-1}} (\mu + \eta^{s'} |Du|)^{s-2} (\mu + |Du|) |v - w| \\
&= \frac{c(n, \ell)}{(r - \rho)} \int_{B_R} \eta^{\frac{1}{s-1}} (\mu + |Dw - D(\eta^{s'})u|)^{s-2} (\mu + |Du|) |v - w| \\
&\leq \frac{c(n, \ell, s)}{(r - \rho)} \int_{B_R} |Dw|^{s-2} (\mu + |Du|) |v - w|
\end{aligned}$$

$$\begin{aligned}
& + \frac{c(n, \ell, s)}{(r - \rho)} \int_{B_R} (\mu + |D(\eta^{s'})u|)^{s-2} (\mu + |Du|) |v - w| \\
& \leq \varepsilon \int_{B_R} |Dv - Dw|^s + \sigma \int_{B_R} |Dw|^s + C(n, \varepsilon, \sigma, \ell, s) \frac{R^s}{(r - \rho)^s} \int_{B_R} |Du|^s \\
& \quad + \frac{C(n, \varepsilon, \sigma, \ell, s)}{(r - \rho)^s} \int_{B_R} |u|^s + C \frac{\mu^s R^{n+s}}{(r - \rho)^s}.
\end{aligned} \tag{3.12}$$

Combining estimates (3.7), (3.8), (3.9), (3.10), (3.11) and (3.12) we conclude that

$$\begin{aligned}
\nu \int_{B_R} |Dv - Dw|^s & \leq \varepsilon (5 + C(n, s)) \int_{B_R} |Dv - Dw|^s + 2\sigma \int_{B_R} |Dw|^s + \frac{c}{(r - \rho)^s} \int_{B_R} |u|^s \\
& \quad + c \left(\int_{B_R} V(x, B) \right)^{\frac{t-s}{t}} \left(\int_{B_R} (\mu + |Dw|^2)^{\frac{t}{2}} \right)^{\frac{s}{t}} \\
& \quad + c \frac{R^s}{(r - \rho)^s} \int_{B_R} |Du|^s + c \frac{d^s k_0^s}{\delta^s (r - \rho)^s} \int_{B_R} |G|^s \\
& \quad + c \int_{B_R} |Du|^s \chi_{B_r \setminus B_\rho} + c \mu^s R^n + C \frac{\mu^s R^{n+s}}{(r - \rho)^s},
\end{aligned}$$

where $c = c(\varepsilon, \sigma, s, n, \ell, L)$. Choosing $\varepsilon = \frac{\nu}{2(5+C(n,s))}$, we can reabsorb the first integral in the right hand side of previous estimate by the left hand side thus obtaining

$$\begin{aligned}
\frac{\nu}{2} \int_{B_R} |Dv - Dw|^s & \leq 2\sigma \int_{B_R} |Dw|^s + \frac{c}{(r - \rho)^s} \int_{B_R} |u|^s \\
& \quad + c \left(\int_{B_R} V(x, B) \right)^{\frac{t-s}{t}} \left(\int_{B_R} (\mu + |Dw|^2)^{\frac{t}{2}} \right)^{\frac{s}{t}} \\
& \quad + c \frac{R^s}{(r - \rho)^s} \int_{B_R} |Du|^s + c \left(\frac{d^s k_0^s}{\delta^s (r - \rho)^s} + 1 \right) \int_{B_R} |G|^s \\
& \quad + c \int_{B_R} |Du|^s \chi_{B_r \setminus B_\rho} + c \mu^s R^n + C \frac{\mu^s R^{n+s}}{(r - \rho)^s},
\end{aligned} \tag{3.13}$$

where $c = c(\nu, \sigma, s, n, \ell, L)$. Consider the ball $B_{\delta R} = B(y, \delta R)$, and observe that

$$\oint_{B_{\delta R}} |Dw - (Dw)_{B_{\delta R}}|^s \leq C(s) \oint_{B_{\delta R}} |Dv - (Dv)_{B_{\delta R}}|^s + C(s) \delta^{-n} \oint_{B_R} |Dw - Dv|^s.$$

Now we can estimate the two terms on the right hand side with the help of estimate (3.13) and Lemma 11 as follows

$$\begin{aligned}
\oint_{B_{\delta R}} |Dw - (Dw)_{B_{\delta R}}|^s & \leq (C(s) \delta^{\alpha s} + 2\sigma \delta^{-n}) \oint_{B_R} |Dw|^s + \frac{c \delta^{-n}}{(r - \rho)^s} \oint_{B_R} |u|^s \\
& \quad + c \delta^{-n} \left(\oint_{B_R} V(x, B) \right)^{\frac{t-s}{t}} \left(\oint_{B_R} (\mu + |Dw|^2)^{\frac{t}{2}} \right)^{\frac{s}{t}} \\
& \quad + c \delta^{-n} \frac{R^s}{(r - \rho)^s} \oint_{B_R} |Du|^s + c \delta^{-n} \left(\frac{d^s k_0^s \delta^{-s}}{(r - \rho)^s} + 1 \right) \oint_{B_R} |G|^s \\
& \quad + c \delta^{-n} \oint_{B_R} |Du|^s \chi_{B_r \setminus B_\rho} + c \mu^s \delta^{-n} + \mu^s \delta^{-n-s} \frac{d^s k_0^s}{(r - \rho)^s}.
\end{aligned}$$

By the classical theory, since $2B_R \subset \Omega$ and u is a local solution, we have that

$$\int_{B_R} |Du|^s \leq \frac{C}{R^s} \int_{B_{2R}} |u|^s + C \int_{B_{2R}} |G|^s$$

and therefore, from $B_{2R} \subset \tilde{\mathbf{B}}_0$ we conclude that

$$\begin{aligned} \int_{B_{\delta R}} |Dw - (Dw)_{B_{\delta R}}|^s &\leq c(\delta^{\alpha s} + \sigma \delta^{-n}) \int_{B_R} |Dw|^s \\ &\quad + c \left(\frac{d^s k_0^s \delta^{-n-s}}{(r-\rho)^s} + \delta^{-n} \right) \int_{B_{2R}} |G \chi_{\tilde{\mathbf{B}}_0}|^s + c \left(\frac{\delta^{-n-s}}{(r-\rho)^s} + \delta^{-n} \right) \int_{B_{2R}} |u \chi_{\tilde{\mathbf{B}}_0}|^s \\ &\quad + c \delta^{-n} \left(\int_{B_R} V(x, B_R) dx \right)^{\frac{t-s}{t}} \left(\int_{B_R} (\mu + |Dw|^2)^{\frac{t}{2}} \right)^{\frac{s}{t}} \\ &\quad + c \delta^{-n} \int_{B_R} |Du|^s \chi_{\mathbf{B}_r \setminus \mathbf{B}_\rho} + c \mu^s \delta^{-n} + \mu^s \delta^{-n-s} \frac{d^s k_0^s}{(r-\rho)^s} \\ &\leq c \delta^{\alpha s} \int_{B_R} |Dw|^s + \left(\frac{c k_0^s d^s \delta^{-n-s}}{(r-\rho)^s} + \delta^{-n} \right) \int_{B_{2R}} |G \chi_{\tilde{\mathbf{B}}_0}|^s + \left(\frac{c \delta^{-n-s}}{(r-\rho)^s} + \delta^{-n} \right) \int_{B_{2R}} |u \chi_{\tilde{\mathbf{B}}_0}|^s \\ &\quad + c \delta^{-n} \left(\int_{B_R} V(x, B_R) dx \right)^{\frac{t-s}{t}} \left(\int_{B_R} (\mu + |Dw|^2)^{\frac{t}{2}} \right)^{\frac{s}{t}} \\ &\quad + c \delta^{-n} \int_{B_R} |Du|^s \chi_{\mathbf{B}_r \setminus \mathbf{B}_\rho} + c \mu^s \delta^{-n} + \mu^s \delta^{-n-s} \frac{d^s k_0^s}{(r-\rho)^s}, \end{aligned}$$

where we chose $\sigma = \delta^{\alpha s+n}$. Now, we take supremum over all possible values $R \in (0, k_0 d/\delta)$, and we get

$$\begin{aligned} \mathcal{M}_{s, k_0 d}^\#(Dw)(y)^s &\leq c \delta^{\alpha s} \mathcal{M}_s(Dw)(y)^s + c \delta^{-n} \mathcal{M}_s(|Du| \chi_{\mathbf{B}_r \setminus \mathbf{B}_\rho})^s(y) \\ &\quad + c \left(\frac{d^s k_0^s \delta^{-n-s}}{(r-\rho)^s} + \delta^{-n} \right) \mathcal{M}_s(G \chi_{\tilde{\mathbf{B}}_0})^s(y) + c \left(\frac{\delta^{-n-s}}{(r-\rho)^s} + \delta^{-n} \right) \mathcal{M}_s(u \chi_{\tilde{\mathbf{B}}_0})^s(y) \\ &\quad + c \delta^{-n} \left(\mathcal{M}_t(Dw)(y)^s \right) \sup_{0 < R < k_0 d/\delta} \left(\int_{B_R} V(x, B_R) dx \right)^{\frac{t-s}{t}} \\ &\quad + c \mu^s \delta^{-n} + \mu^s \delta^{-n-s} \frac{d^s k_0^s}{(r-\rho)^s}. \end{aligned}$$

Now, we raise to the power $\frac{q}{s}$, and then integrate with respect to y over $k_0 \mathbf{B}_0$. We obtain

$$\begin{aligned} \|\mathcal{M}_{s, k_0 d}^\#(Dw)\|_{L^q(k_0 \mathbf{B}_0)}^q &\leq C(s, q) \delta^{\alpha q} \|\mathcal{M}_s(Dw)\|_{L^q(\mathbb{R}^n)}^q + c \delta^{\frac{-nq}{s}} \|\mathcal{M}_s(|Du| \chi_{\mathbf{B}_r \setminus \mathbf{B}_\rho})\|_{L^q(\mathbb{R}^n)}^q \\ &\quad + c \left(\frac{k_0^q d^q \delta^{\frac{-nq}{s}-q}}{(r-\rho)^q} + \delta^{\frac{-nq}{s}} \right) \|\mathcal{M}_s(G \chi_{\tilde{\mathbf{B}}_0})\|_{L^q(\mathbb{R}^n)}^q + c \left(\frac{\delta^{\frac{-nq}{s}-q}}{(r-\rho)^q} + \delta^{\frac{-nq}{s}} \right) \|\mathcal{M}_s(u \chi_{\tilde{\mathbf{B}}_0})\|_{L^q(\mathbb{R}^n)}^q \\ &\quad + c \delta^{\frac{-nq}{s}} \left(\|\mathcal{M}_t(Dw)\|_{L^q(\mathbb{R}^n)}^q \right) \sup_{y \in k_0 \mathbf{B}_0} \sup_{0 < R < k_0 d/\delta} \left(\int_{B_R} V(x, B_R) dx \right)^{\frac{t-s}{t}} \\ &\quad + c \mu^q |k_0 \mathbf{B}_0| \left(\delta^{\frac{-nq}{s}} + \frac{\delta^{\frac{-nq}{s}-q} d^q}{(r-\rho)^s} \right), \end{aligned}$$

where $c = c(n, s, q, \ell, L, \nu, \delta)$. Now we use Lemma 15 (i) and (ii), and obtain

$$\begin{aligned}
\|Dw\|_{L^q(\mathbf{B}_0)}^q &\leq C(n, s, q) \delta^{\alpha q} \|Dw\|_{L^q(\mathbf{B}_0)}^q + c \delta^{\frac{-nq}{s}} \|Du\chi_{\mathbf{B}_r \setminus \mathbf{B}_\rho}\|_{L^q(\mathbb{R}^n)}^q \\
&+ c \left(\frac{d^q k_0^q \delta^{\frac{-nq}{s} - q}}{(r - \rho)^q} + \delta^{\frac{-nq}{s}} \right) \|G\chi_{\tilde{\mathbf{B}}_0}\|_{L^q(\mathbb{R}^n)}^q + c \left(\frac{\delta^{\frac{-nq}{s} - q}}{(r - \rho)^q} + \delta^{\frac{-nq}{s}} \right) \|u\chi_{\tilde{\mathbf{B}}_0}\|_{L^q(\mathbb{R}^n)}^q \\
&+ c \delta^{\frac{-nq}{s}} \left(\|Dw\|_{L^q(\mathbf{B}_0)}^q \right) \sup_{y \in k_0 \mathbf{B}_0} \sup_{0 < R < k_0 d / \delta} \left(\int_{B_R} V(x, B_R) dx \right)^{\frac{t-s}{t}} \\
&+ c \mu^q |k_0 \mathbf{B}_0| \left(\delta^{\frac{-nq}{s}} + \frac{\delta^{\frac{-nq}{s} - q} d^q}{(r - \rho)^s} \right). \tag{3.14}
\end{aligned}$$

Our next aim consists of inserting the two terms with Dw on the right hand side into the term on the left hand side, by making their coefficients as small as possible. To do this, we first look at the term $C(n, s, q) \delta^{\alpha q} \|Dw\|_{L^q(\mathbf{B}_0)}^q$. To be absorbed on the left hand side, it suffices to choose δ such that

$$C(n, s, q) \delta^{\alpha q} = \frac{1}{4} \iff \delta = \frac{1}{[4C(n, s, q)]^{\frac{1}{\alpha q}}}.$$

Note that this choice of $\delta = \delta(n, s, q, \alpha) > 0$ is independent of d . Therefore, taking into account that δ has been fixed, estimate (3.14) becomes

$$\begin{aligned}
\|Dw\|_{L^q(\mathbf{B}_0)}^q &\leq c \|Du\chi_{\mathbf{B}_r \setminus \mathbf{B}_\rho}\|_{L^q(\mathbb{R}^n)}^q \\
&+ c \left(\frac{k_0^q d^q}{(r - \rho)^q} + 1 \right) \|G\chi_{\tilde{\mathbf{B}}_0}\|_{L^q(\mathbb{R}^n)}^q + \left(\frac{c}{(r - \rho)^q} + 1 \right) \|u\chi_{\tilde{\mathbf{B}}_0}\|_{L^q(\mathbb{R}^n)}^q \\
&+ \tilde{c} \left(\|Dw\|_{L^q(\mathbf{B}_0)}^q \right) \sup_{y \in k_0 \mathbf{B}_0} \sup_{0 < R < k_0 d / \delta} \left(\int_{B_R} V(x, B_R) dx \right)^{\frac{t-s}{t}} \\
&+ c \mu^q |k_0 \mathbf{B}_0| \frac{d^q}{(r - \rho)^q}, \tag{3.15}
\end{aligned}$$

with constants c and \tilde{c} depending on n, s, q, ℓ, L, ν but independent of d . Now, if $k_0 d < \frac{d(x_0, \partial\Omega)}{2}$ then

$$\sup_{y \in k_0 \mathbf{B}_0} \sup_{0 < R < k_0 d / \delta} \left(\int_{B_R} V(x, B_R) dx \right)^{\frac{t-s}{t}} \leq \sup_{y \in B(x_0, \frac{d(x_0, \partial\Omega)}{2})} \sup_{0 < R < k_0 d / \delta} \left(\int_{B_R} V(x, B_R) dx \right)^{\frac{t-s}{t}}$$

and moreover, from (3.2) we have that

$$\lim_{d \rightarrow 0} \sup_{y \in B(x_0, \frac{d(x_0, \partial\Omega)}{2})} \sup_{0 < R < k_0 d / \delta} \left(\int_{B_R} V(x, B_R) dx \right)^{\frac{t-s}{t}} = 0.$$

In particular, $d > 0$ can be chosen small enough so that

$$\tilde{c} \sup_{y \in k_0 \mathbf{B}_0} \sup_{0 < R < k_0 d / \delta} \left(\int_{B_R} V(x, B_R) dx \right)^{\frac{t-s}{t}} < \frac{1}{4}.$$

Note that the chosen value d certainly depends on $d(x_0, \partial\Omega)$, \mathbf{A} , ν , ℓ , L , s , q and t . Nevertheless, this allows us to insert the remaining term with Dw into the left hand side. One then gets immediately from (3.14) and our choice of w that

$$\begin{aligned}
\int_{\mathbf{B}_\rho} |Du|^q &\leq 2^n \int_{\mathbf{B}_r} |Dw|^q \leq c \int_{\mathbf{B}_r \setminus \mathbf{B}_\rho} |Du|^q + c \left(\frac{d^q}{(r - \rho)^q} + 1 \right) \int_{\tilde{\mathbf{B}}_0} |G|^q \\
&+ c \left(\frac{1}{(r - \rho)^q} + 1 \right) \int_{\tilde{\mathbf{B}}_0} |u|^q + c \mu^q |\tilde{\mathbf{B}}_0| \left(\frac{d^q}{(r - \rho)^q} + 1 \right).
\end{aligned}$$

Filling the hole, i.e. adding to both sides of previous inequality the quantity

$$c \int_{\mathbf{B}_\rho} |Du|^q$$

we obtain

$$\begin{aligned} \int_{\mathbf{B}_\rho} |Du|^q &\leq \vartheta \int_{\mathbf{B}_r} |Du|^q + c \left(\frac{d^q}{(r-\rho)^q} + 1 \right) \int_{\tilde{\mathbf{B}}_0} |G|^q \\ &\quad + c \left(\frac{1}{(r-\rho)^q} + 1 \right) \int_{\tilde{\mathbf{B}}_0} |u|^q + c \mu^q |\tilde{\mathbf{B}}_0| \left(\frac{d^q}{(r-\rho)^q} + 1 \right), \end{aligned}$$

with $0 < \vartheta < 1$. Since the above estimate is valid for arbitrary radii $\frac{d}{2} < \rho < r < d$, by virtue of Lemma 12, we conclude that

$$\oint_{\frac{1}{2}\mathbf{B}_0} |Du|^q \leq C(n, \nu, \ell, L, s, q) \left(\mu^q + \oint_{\tilde{\mathbf{B}}_0} |G|^q + \frac{1}{d^q} \oint_{\tilde{\mathbf{B}}_0} |u|^q \right).$$

Since $\tilde{\mathbf{B}}_0 = (1 + \frac{2}{\delta})k_0\mathbf{B}_0$, the claim follows by simply choosing $\lambda = 2(1 + \frac{2}{\delta})k_0$. \square

We are now in a position to give the

Proof of Theorem 5. Fix a ball $B_R(x_0) \Subset \Omega$ with $0 < R < \lambda d_0$ where λ and d_0 are the ones determined in Theorem 16. Moreover fix a smooth kernel $\phi \in C_c^\infty(B_1(0))$ with $\phi \geq 0$ and $\int_{B_1(0)} \phi = 1$, let us consider the corresponding family of mollifiers $(\phi_\varepsilon)_{\varepsilon>0}$ and put

$$\mathbf{A}_\varepsilon(x, \xi) := \mathbf{A}(\cdot, \xi) * \phi_\varepsilon(x) = \int_{B_1} \phi(\omega) \mathbf{A}(x - \varepsilon\omega, \xi) d\omega \quad (3.16)$$

and

$$G_\varepsilon = G * \phi_\varepsilon \quad (3.17)$$

each positive $\varepsilon < \text{dist}(B_R, \partial\Omega)$. One can easily check that the assumptions **(A1)**, **(A2)**, **(A3)** imply

$$(H1) \quad \langle \mathbf{A}_\varepsilon(x, \xi) - \mathbf{A}_\varepsilon(x, \eta), \xi - \eta \rangle \geq \nu(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{s-2}{2}} |\eta - \xi|^2$$

$$(H2) \quad |\mathbf{A}_\varepsilon(x, \xi) - \mathbf{A}_\varepsilon(x, \eta)| \leq L|\xi - \eta|(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{s-2}{2}}$$

$$(H3) \quad |\mathbf{A}_\varepsilon(x, \xi)| \leq \ell(\mu^2 + |\xi|^2)^{\frac{s-1}{2}}$$

for almost every $x \in \Omega$ and for all $\xi, \eta \in \mathbb{R}^n$. Moreover, setting

$$V_\varepsilon(x, B_R) = \sup_{\xi \neq 0} \frac{|\mathbf{A}_\varepsilon(x, \xi) - \mathbf{A}_{\varepsilon, B_R}(\xi)|}{(\mu^2 + |\xi|^2)^{\frac{s-1}{2}}} \quad \text{with} \quad \mathbf{A}_{\varepsilon, B_R}(\xi) = \oint_{B_R} \mathbf{A}_\varepsilon(y, \xi) dy$$

since $x \rightarrow \mathbf{A}_\varepsilon(x, \xi)$ is \mathcal{C}^∞ smooth, we have that

$$(H4) \quad \lim_{r \rightarrow 0} \sup_{r(B) < r} \sup_{c(B) \in B_R} \oint_B V_\varepsilon(x, B) dx = 0.$$

For further needs we record that, since $\mathbf{A}_\varepsilon(x, Du) \in L^{\frac{s}{s-1}}(B_R)$, that

$$\mathbf{A}_\varepsilon(x, Du) \rightarrow \mathbf{A}(x, Du) \quad \text{strongly in } L^{\frac{s}{s-1}}(B_R) \quad (3.18)$$

and also that, since $G \in L^q(B_R)$,

$$G_\varepsilon \rightarrow G \quad \text{strongly in } L^q_{\text{loc}}(B_R). \quad (3.19)$$

Let $u \in W^{1,s}_{\text{loc}}(\Omega)$ be a solution of the equation (1.1) and let us denote by $u_\varepsilon \in W^{1,s}(B_R)$ the unique solution of the Dirichlet problem

$$(P_\varepsilon) \quad \begin{cases} \operatorname{div} \mathbf{A}_\varepsilon(x, Du_\varepsilon) = \operatorname{div}(|G_\varepsilon|^{s-1} G_\varepsilon) & \text{in } B_R \\ u_\varepsilon = u & \text{on } \partial B_R \end{cases}$$

By the classical theory, since $x \rightarrow \mathbf{A}_\varepsilon(x, \xi)$ is C^∞ smooth, we have that $Du_\varepsilon \in L^q$, for every $q \geq s$.

Using $\varphi = u_\varepsilon - u$ as test function in the equation (P_ε) and in the equation (1.1), we have

$$\begin{aligned} \int_{B_R} \langle \mathbf{A}_\varepsilon(x, Du_\varepsilon), Du - Du_\varepsilon \rangle dx &= \int_{B_R} |G_\varepsilon|^{s-1} \langle G_\varepsilon, Du - Du_\varepsilon \rangle dx \\ \int_{B_R} \langle \mathbf{A}(x, Du), Du - Du_\varepsilon \rangle dx &= \int_{B_R} |G|^{s-1} \langle G, Du - Du_\varepsilon \rangle dx \end{aligned}$$

Subtracting the second equation from the first one, we obtain

$$\int_{B_R} \langle \mathbf{A}_\varepsilon(x, Du_\varepsilon) - \mathbf{A}(x, Du), Du - Du_\varepsilon \rangle dx = \int_{B_R} \langle |G_\varepsilon|^{s-1} G_\varepsilon - |G|^{s-1} G, Du - Du_\varepsilon \rangle dx \quad (3.20)$$

Inequality (H1) yields

$$\begin{aligned} & \nu \int_{B_R} (\mu^2 + |Du|^2 + |Du_\varepsilon|^2)^{\frac{s-2}{2}} |Du - Du_\varepsilon|^2 dx \\ & \leq \int_{B_R} \langle \mathbf{A}_\varepsilon(x, Du_\varepsilon) - \mathbf{A}_\varepsilon(x, Du), Du - Du_\varepsilon \rangle dx \\ & = \int_{B_R} \langle \mathbf{A}(x, Du) - \mathbf{A}_\varepsilon(x, Du), Du - Du_\varepsilon \rangle dx \\ & + \int_{B_R} \langle |G_\varepsilon|^{s-1} G_\varepsilon - |G|^{s-1} G, Du - Du_\varepsilon \rangle dx \\ & \leq \left(\int_{B_R} |\mathbf{A}(x, Du) - \mathbf{A}_\varepsilon(x, Du)|^{\frac{s}{s-1}} dx \right)^{\frac{s-1}{s}} \left(\int_{B_R} |Du - Du_\varepsilon|^s dx \right)^{\frac{1}{s}} \\ & + \left(\int_{B_R} |G_\varepsilon - G|^s dx \right)^{\frac{s-1}{s}} \left(\int_{B_R} |Du - Du_\varepsilon|^s dx \right)^{\frac{1}{s}}, \end{aligned} \quad (3.21)$$

where we used the equality (3.20) and Hölder's inequality. Since $s \geq 2$, by well known means, from estimate (3.21) we deduce

$$\int_{B_R} |Du - Du_\varepsilon|^s dx \leq c \int_{B_R} |\mathbf{A}(x, Du) - \mathbf{A}_\varepsilon(x, Du)|^{\frac{s}{s-1}} dx + \int_{B_R} |G_\varepsilon - G|^s dx.$$

Taking the limit as $\varepsilon \rightarrow 0$ in previous inequality and recalling (3.18) and (3.19), we deduce that u_ε converges strongly to u in $W^{1,s}$. Since the operator A_ε satisfies estimates (H1)–(H4) and $Du_\varepsilon \in L^q$ for every $q \geq s$, we are legitimate to apply the a priori estimate of Theorem 16 to each u_ε thus getting

$$\int_{B_\rho} |Du_\varepsilon|^q \leq C \left(\mu^q + \int_{B_{\lambda\rho}} |u_\varepsilon|^q + \int_{B_{\lambda\rho}} |G_\varepsilon|^q \right) \quad (3.22)$$

for every $q > s$ and for every positive ρ such that $B_{\lambda\rho} \subset B_R$. Let us define the decreasing sequence of exponents

$$\begin{cases} q_0 = q \\ q_j = \frac{nq_{j-1}}{n+q_{j-1}} \quad j \in \mathbb{N} \end{cases}$$

Note that, since $q_j \searrow 0$, there exists $h \in \mathbb{N}$ such that $q_h \leq s^*$. Chose now $\rho = \rho_h$ so small to have $\lambda^h \rho < R$ and let $r_i = \lambda^i \rho$. Since $G \in L^q(B_R)$ we have $G \in L^{q_i}(B_R)$ for every $i \in \mathbb{N}$ and so we can write inequality (3.22) as follows

$$\begin{aligned} \int_{B_{r_i}} |Du_\varepsilon|^{q_i} &\leq \frac{C_{q_i}}{r_{i+1}^{q_i}} \int_{B_{r_{i+1}}} |u_\varepsilon|^{q_i} + C_{q_i} \int_{B_{r_{i+1}}} |G_\varepsilon|^{q_i} + C_{q_i} \mu^{q_i} |B_{r_{i+1}}| \\ &\leq \frac{C_{q_i}}{r_{i+1}^{q_i}} \left(\int_{B_{r_{i+1}}} |u_\varepsilon|^{q_{i+1}} + |Du_\varepsilon|^{q_{i+1}} \right)^{\frac{q_i}{q_{i+1}}} + C_{q_i} \int_{B_{r_{i+1}}} |G_\varepsilon|^{q_i} + C_{q_i} \mu^{q_i} |B_{r_{i+1}}| \\ &\leq \frac{C_{q_i}}{r_{i+1}^{q_i}} \left[\int_{B_{r_{i+1}}} |u_\varepsilon|^{q_{i+1}} + \frac{C_{q_{i+1}}}{r_{i+2}^{q_{i+1}}} \int_{B_{r_{i+2}}} |u_\varepsilon|^{q_{i+1}} + C_{q_{i+1}} \int_{B_{r_{i+2}}} |G_\varepsilon|^{q_{i+1}} + C_{q_{i+1}} \mu^{q_{i+1}} |B_{r_{i+2}}| \right]^{\frac{q_i}{q_{i+1}}} \\ &\quad + C_{q_i} \int_{B_{r_{i+1}}} |G_\varepsilon|^{q_i} + C_{q_i} \mu^{q_i} |B_{r_{i+1}}| \\ &\leq \frac{C_{q_i} C_{q_{i+1}}}{(r_{i+1} r_{i+2})^{q_i}} \left(\int_{B_{r_{i+2}}} |u_\varepsilon|^{q_{i+1}} \right)^{\frac{q_i}{q_{i+1}}} + \frac{C_{q_i} C_{q_{i+1}}}{(r_{i+1})^{q_i}} \left(\int_{B_{r_{i+2}}} |G_\varepsilon|^{q_{i+1}} \right)^{\frac{q_i}{q_{i+1}}} + C_{q_i} \int_{B_{r_{i+2}}} |G_\varepsilon|^{q_i} \\ &\quad + C_{q_i} \mu^{q_i} |B_{r_{i+1}}| + C_{q_{i+1}} C_{q_i} \mu^{q_i} |B_{r_{i+1}}| |B_{r_{i+2}}|^{\frac{q_i}{q_{i+1}}} \end{aligned} \quad (3.23)$$

where we used first Sobolev inequality and again inequality at (3.22) and finally Young's inequality. Iterating estimate (3.23), from $i = 0$ to $i = h - 1$, we deduce that

$$\int_{B_\rho} |Du_\varepsilon|^q \leq \tilde{C}_h \left(\int_{B_{\lambda^h \rho}} |u_\varepsilon|^{q_h} \right)^{\frac{q}{q_h}} + \tilde{C}_h \int_{B_R} |G_\varepsilon|^q + \tilde{C}_h \mu^q,$$

where $\tilde{C}_h = \prod_{i=0}^{h-1} \frac{C_{q_i}}{r_{i+1}^{q_i}}$. Since $q_h \leq s^*$, by virtue of the strong convergence of u_ε to u in $W^{1,s}$, we can pass to limit as $\varepsilon \rightarrow 0$ in previous estimate to deduce

$$\int_{B_\rho} |Du|^q \leq \tilde{C}_h \left(\int_{B_R} |u|^{q_h} \right)^{\frac{q}{q_h}} + \tilde{C}_h \int_{B_R} |G|^q + \tilde{C}_h \mu^q,$$

i.e. the conclusion. □

4 Proof of Theorem 1

We first prove that if (1.3) is satisfied then \mathcal{A} has the locally uniform VMO property (3.2).

Lemma 17. *Let \mathcal{A} be such that $(\mathcal{A}1), (\mathcal{A}2), (\mathcal{A}3)$ hold. Assume that (1.3) is satisfied. Then \mathcal{A} is locally uniformly in VMO, that is, (3.2) holds with $s = 2$.*

Proof. We have

$$\begin{aligned}
\int_B V(x, B) dx &= \int_B \sup_{\xi \neq 0} \frac{|\mathcal{A}(x, \xi) - \mathcal{A}_B(\xi)|}{(\mu^2 + |\xi|^2)^{\frac{1}{2}}} dx \\
&\leq \int_B \sup_{\xi \neq 0} \int_B \frac{|\mathcal{A}(x, \xi) - \mathcal{A}(y, \xi)|}{(\mu^2 + |\xi|^2)^{\frac{1}{2}}} dy dx \\
&\leq \int_B \sup_{\xi \neq 0} \int_B (g(x) + g(y)) |x - y|^\alpha dy dx \\
&= \int_B \int_B (g(x) + g(y)) |x - y|^\alpha dy dx \\
&\leq \left(\int_B \int_B (g(x) + g(y))^{\frac{n}{\alpha}} dy dx \right)^{\frac{\alpha}{n}} \left(\int_B \int_B |x - y|^{\frac{n\alpha}{n-\alpha}} dy dx \right)^{\frac{n-\alpha}{n}} \\
&\leq \left(\frac{1}{|B|} \int_B g^{\frac{n}{\alpha}} \right)^{\frac{\alpha}{n}} C(\alpha, n) |B|^{\frac{\alpha}{n}} = C(n, \alpha) \int_B g^{\frac{n}{\alpha}}
\end{aligned}$$

and thus (3.2) holds. \square

Proof of Theorem 1. Given a test function $\varphi \in \mathcal{C}_c^\infty(\Omega)$ such that $\text{supp } \tau_{-h}\varphi \subset \Omega$, we test the equation

$$\text{div } \mathcal{A}(x, Du) = 0$$

with φ and $\tau_{-h}\varphi$, and combine the resulting identities. We have

$$\int \langle \mathcal{A}(x+h, Du(x+h)) - \mathcal{A}(x+h, Du), \nabla \varphi \rangle = - \int \langle \mathcal{A}(x+h, Du(x)) - \mathcal{A}(x, Du(x)), \nabla \varphi \rangle.$$

Now, by setting

$$\mathcal{A}_h(x, \xi) = \frac{1}{|h|^\alpha} (\mathcal{A}(x+h, |h|^\alpha \xi + Du(x)) - \mathcal{A}(x+h, Du))$$

and $v_h = \frac{\Delta_h u}{|h|^\alpha}$, we immediately see that v_h is a weak solution of

$$\text{div } \mathcal{A}_h(x, Dv_h) = \text{div } G_h \tag{4.1}$$

where

$$G_h(x) = -\frac{1}{|h|^\alpha} (\mathcal{A}(x+h, Du(x)) - \mathcal{A}(x, Du(x))). \tag{4.2}$$

It is immediate to check that the new \mathcal{A}_h still satisfies (A1), (A2) with the same constants of \mathcal{A} . Moreover, (A3) is also satisfied by \mathcal{A}_h but now with $\mu = 0$. We also note that

$$|G_h(x)| = \left| \frac{\mathcal{A}(x+h, Du(x)) - \mathcal{A}(x, Du(x))}{|h|^\alpha} \right| \leq (g(x+h) + g(x)) (\mu^2 + |Du(x)|^2)^{\frac{1}{2}},$$

Now, we know from Lemma 17 that \mathcal{A} is locally uniformly in VMO , and so Theorem 5 ensures that $Du \in L_{loc}^r$ for each finite $r > 2$. In particular, if $2 \leq p < \frac{n}{\alpha}$ then $Du \in L_{loc}^{p_\alpha^*}$ and as a consequence $G_h \in L_{loc}^p$. It then follows that Lemma 14 can be applied to (4.1) with $\mu = 0$ and so there exists $p_0 = p_0(n, \nu, \ell) > 2$ such that if one further has $2 \leq p < p_0$ then

$$\|Dv_h\|_{L^p(B)} \leq C_0 \left(\frac{1}{r_B} \|v_h\|_{L^p(2B)} + \|G_h\|_{L^p(2B)} \right) \tag{4.3}$$

for each ball B with radius r_B such that $2B \subset \Omega$. In terms of u , this reads as

$$\begin{aligned} \left\| \frac{\Delta_h(Du)}{|h|^\alpha} \right\|_{L^p(B)} &\leq C_0 \left(\frac{1}{r_B} \left\| \frac{\Delta_h u}{|h|^\alpha} \right\|_{L^p(2B)} + \|G_h\|_{L^p(2B)} \right) \\ &\leq C_0 \left(\frac{1}{r_B} \left\| \frac{\Delta_h u}{|h|^\alpha} \right\|_{L^p(2B)} + \|g\|_{L^{\frac{n}{\alpha}}(2B)} \|(1 + |Du|^2)^{\frac{1}{2}}\|_{L^{\frac{np}{n-\alpha p}}(2B)} \right) \end{aligned}$$

and so taking supremum for $|h| < \delta$, $\delta > 0$ small enough,

$$\sup_h \left\| \frac{\Delta_h(Du)}{|h|^\alpha} \right\|_{L^p(B)} \leq C_0 \left(\frac{1}{r_B} \sup_h \left\| \frac{\Delta_h u}{|h|^\alpha} \right\|_{L^p(2B)} + \|g\|_{L^{\frac{n}{\alpha}}(2B)} \|(1 + |Du|^2)^{\frac{1}{2}}\|_{L^{\frac{np}{n-\alpha p}}(2B)} \right)$$

We now use Lemma 7 to see that the term $\sup_h \left\| \frac{\Delta_h u}{|h|^\alpha} \right\|_{L^p(2B)}$ is finite, since $u \in W_{loc}^{1,p}$. We then obtain that $Du \in B_{p,\infty,loc}^\alpha$, as claimed. When \mathcal{A} is linear in the gradient variable, that is $\mathcal{A}(x, \xi) = A(x)\xi$, one immediately sees that $x \mapsto \mathcal{A}_h(x, \xi)$ is locally uniformly in VMO , and therefore the restriction $p < p_0$ at (4.3) is not needed. \square

5 Proof of Theorems 2, 3 and 4

We first prove that if \mathcal{A} satisfies (A1), (A2), (A3), (A4) then it is locally uniformly in VMO . When \mathcal{A} is linear in the second variable, this comes from Lemma 6.

Lemma 18. *Let \mathcal{A} be such that (A1), (A2), (A3), (A4) hold. Then \mathcal{A} is locally uniformly in VMO , that is, (3.2) holds with $s = 2$.*

Proof. Given a point $x \in \Omega$, let us write $A_k(x) = \{y \in \Omega : 2^{-k} \leq |x - y| < 2^{-k+1}\}$. We have

$$\begin{aligned} \int_B V(x, B) dx &= \int_B \sup_{\xi \neq 0} \frac{|\mathcal{A}(x, \xi) - \mathcal{A}_B(\xi)|}{(\mu^2 + |\xi|^2)^{\frac{1}{2}}} dx \\ &\leq \int_B \sup_{\xi \neq 0} \int_B \frac{|\mathcal{A}(x, \xi) - \mathcal{A}(y, \xi)|}{(\mu^2 + |\xi|^2)^{\frac{1}{2}}} dy dx \\ &= \int_B \sup_{\xi \neq 0} \frac{1}{|B|} \sum_k \int_{B \cap A_k(x)} \frac{|\mathcal{A}(x, \xi) - \mathcal{A}(y, \xi)|}{(\mu^2 + |\xi|^2)^{\frac{1}{2}}} dy dx \\ &\leq \frac{1}{|B|^2} \sum_k \int_B \int_{B \cap A_k(x)} |x - y|^\alpha (g_k(x) + g_k(y)) dy dx \end{aligned}$$

The last term above is bounded by

$$\left(\frac{1}{|B|^2} \sum_k \int_B \int_{B \cap A_k(x)} |x - y|^{\frac{n\alpha}{n-\alpha}} dy dx \right)^{\frac{n-\alpha}{n}} \left(\frac{1}{|B|^2} \sum_k \int_B \int_{B \cap A_k(x)} (g_k(x) + g_k(y))^{\frac{n}{\alpha}} dy dx \right)^{\frac{\alpha}{n}} = I \cdot II$$

The first sum is very easy to handle, since

$$I = \left(\frac{1}{|B|^2} \sum_k \int_B \int_{B \cap A_k(x)} |x - y|^{\frac{n\alpha}{n-\alpha}} dy dx \right)^{\frac{n-\alpha}{n}} \leq C(n, \alpha) |B|^{\frac{\alpha}{n}}$$

Concerning the second, we see that

$$\begin{aligned}
II &\leq \left(\frac{1}{|B|^2} \sum_k |B \cap A_k(x)| \int_B g_k(x)^{\frac{n}{\alpha}} dx \right)^{\frac{\alpha}{n}} \\
&\leq \left(\frac{1}{|B|^2} \sum_k \left(\int_B g_k(x)^{\frac{n}{\alpha}} dx \right)^{\frac{\alpha q}{n}} \right)^{\frac{\alpha}{n} \frac{n}{\alpha q}} \left(\frac{1}{|B|^2} \sum_k |B \cap A_k(x)|^{\frac{\alpha q}{\alpha q - n}} \right)^{\frac{\alpha}{n} \frac{\alpha q - n}{\alpha q}} \\
&= \frac{1}{|B|^{\frac{2}{q}}} \left(\sum_k \|g_k\|_{L^{\frac{n}{\alpha}}(B)}^q \right)^{\frac{1}{q}} \frac{1}{|B|^{2(\frac{\alpha}{n} - \frac{1}{q})}} \left(\sum_k |B \cap A_k(x)|^{\frac{\alpha q}{\alpha q - n}} \right)^{\frac{\alpha}{n} \frac{\alpha q - n}{\alpha q}} \\
&\leq \frac{1}{|B|^{\frac{2}{q}}} \left(\sum_k \|g_k\|_{L^{\frac{n}{\alpha}}(B)}^q \right)^{\frac{1}{q}} \frac{1}{|B|^{2(\frac{\alpha}{n} - \frac{1}{q})}} C(n, \alpha, q) |B|^{\frac{\alpha}{n}} = C(n, \alpha, q) |B|^{-\frac{\alpha}{n}} \left(\sum_k \|g_k\|_{L^{\frac{n}{\alpha}}(B)}^q \right)^{\frac{1}{q}}
\end{aligned}$$

thus

$$\int_B V(x, B) dx \leq I \cdot II \leq C(n, \alpha, q) \left(\sum_k \|g_k\|_{L^{\frac{n}{\alpha}}(B)}^q \right)^{\frac{1}{q}}.$$

In order to get the *VMO* condition, it just remains to prove that

$$\lim_{r \rightarrow 0} \sup_{x \in K} \left(\sum_k \|g_k\|_{L^{\frac{n}{\alpha}}(B(x, r))}^q \right)^{\frac{1}{q}} = 0$$

on every compact set $K \subset \Omega$. To do this, we fix $r > 0$ small enough, and observe that the function $x \mapsto \|g_k\|_{\ell^q(L^{\frac{n}{\alpha}}(B(x, r)))}$ is continuous on the set $\{x \in \Omega : d(x, \partial\Omega) > r\}$, as a uniformly converging series of continuous functions. As a consequence, there is a point $x_r \in K$ (at least for small enough $r > 0$) such that

$$\sup_{x \in K} \|g_k\|_{\ell^q(L^{\frac{n}{\alpha}}(B(x, r)))} = \|g_k\|_{\ell^q(L^{\frac{n}{\alpha}}(B(x_r, r)))}.$$

Now, from $\|g_k\|_{L^{\frac{n}{\alpha}}(B(x, r))} \leq \|g_k\|_{L^{\frac{n}{\alpha}}(B(x_r, r))}$ and this belongs to ℓ^q , we can use dominated convergence to say that

$$\lim_{r \rightarrow 0} \|g_k\|_{\ell^q(L^{\frac{n}{\alpha}}(B(x_r, r)))} = \left(\sum_k \lim_{r \rightarrow 0} \left(\int_{B(x_r, r)} g_k^{\frac{n}{\alpha}} \right)^{\frac{q\alpha}{n}} \right)^{\frac{1}{q}}.$$

Each of the limits on the term on the right hand side are equal to 0, since the points x_r cannot escape from the compact set K as $r \rightarrow 0$. This finishes the proof. \square

We now prove Theorem 3.

Proof of Theorem 3. Given a test function $\varphi \in \mathcal{C}_c^\infty(\Omega)$ such that $\text{supp } \tau_{-h}\varphi \subset \Omega$, we test the equation with φ and $\tau_{-h}\varphi$, and combine the resulting identities. We have

$$\begin{aligned}
\int \langle \mathcal{A}(x+h, Du(x+h)) - \mathcal{A}(x+h, Du), \nabla \varphi \rangle &= \\
&= \int \langle \Delta_h G, \nabla \varphi \rangle - \int \langle \mathcal{A}(x+h, Du(x)) - \mathcal{A}(x, Du(x)), \nabla \varphi \rangle.
\end{aligned}$$

Now, by setting

$$\mathcal{A}_h(x, \xi) = \frac{1}{|h|^\alpha} (\mathcal{A}(x+h, |h|^\alpha \xi + Du(x)) - \mathcal{A}(x+h, Du))$$

and $v_h = \frac{\Delta_h u}{|h|^\alpha}$, we immediately see that v_h is a weak solution of

$$\operatorname{div} \mathcal{A}_h(x, Dv_h) = \operatorname{div} G_h \quad (5.1)$$

where

$$G_h(x) = \frac{1}{|h|^\alpha} \Delta_h G(x) - \frac{1}{|h|^\alpha} (\mathcal{A}(x+h, Du(x)) - \mathcal{A}(x, Du(x))) \quad (5.2)$$

As before, \mathcal{A}_h still satisfies (A1), (A2), (A3) with same constants ν , L , ℓ but now $\mu = 0$. We also note that, by virtue of (A4) and the assumption on G , we have $G_h \in L^p_{loc}$ for almost every h . Indeed, this is clear for the first term at (5.2), since by assumption $G \in B^{\alpha}_{p,q,loc}$. On the other hand, (A4) tells us that

$$\left| \frac{\mathcal{A}(x+h, Du(x)) - \mathcal{A}(x, Du(x))}{|h|^\alpha} \right| \leq (g_k(x+h) + g_k(x)) (\mu^2 + |Du(x)|^2)^{\frac{1}{2}}, \quad \text{if } 2^{-k} \leq |h| < 2^{-k+1}.$$

Above, $g_k \in L^{\frac{n}{\alpha}}$ by assumption. Also, $(1 + |Du(x)|^2)^{\frac{1}{2}} \in L^{p^*_{loc}}$. To see this, use Lemma 6 with $p < \frac{n}{\alpha}$ and $q \leq p^*_\alpha$ to see that $G \in L^{p^*_{loc}}$, and deduce then that $Du \in L^{p^*_{loc}}$ from Theorem 5 (if $p^*_\alpha \geq 2$) or Lemma 14 (if $p^*_\alpha < 2$ we still have $p'_0 < p < p^*_\alpha$). Hence, we obtain that $G_h \in L^p_{loc}$.

We can use now Lemma 14 at (5.1). If B is a ball with $(2 + |h|)B \subset \Omega$,

$$\|Dv_h\|_{L^p(B)} \leq C_0 \left(\frac{1}{r_B} \|v_h\|_{L^p(2B)} + \|G_h\|_{L^p(2B)} \right), \quad p'_0 < p < p_0 \quad (5.3)$$

where r_B denotes the radius of B , p_0 is as in Lemma 14, and the constant $C_0 = C_0(n, p, \nu, L, s)$ does not depend on h . We now write the above inequality in terms of u , and then take L^q norm with the measure $\frac{dh}{|h|^n}$ restricted to the ball $B(0, R)$ on the h -space. We obtain that

$$\left\| \frac{\Delta_h Du}{|h|^\alpha} \right\|_{L^q(\frac{dh}{|h|^n}; L^p(B))} \leq C_0 \left(\frac{1}{r_B} \left\| \frac{\Delta_h u}{|h|^\alpha} \right\|_{L^q(\frac{dh}{|h|^n}; L^p(2B))} + \|G_h\|_{L^q(\frac{dh}{|h|^n}; L^p(2B))} \right).$$

Above, the first term on the right hand side is finite, since $Du \in L^{p^*_{loc}}$. In order to estimate the last term, we write

$$\|G_h\|_{L^q(\frac{dh}{|h|^n}; L^p(2B))} \leq \left\| \frac{\Delta_h G}{|h|^\alpha} \right\|_{L^q(\frac{dh}{|h|^n}; L^p(2B))} + \left\| \frac{\mathcal{A}(\cdot + h, Du) - \mathcal{A}(\cdot, Du)}{|h|^\alpha} \right\|_{L^q(\frac{dh}{|h|^n}; L^p(2B))}$$

Above, the first term on the right hand side is finite, since by assumption $G \in B^{\alpha}_{p,q,loc}$. Concerning the second term, denote $r_k = 2^{-k} R$. We write the L^q norm in polar coordinates, so $h \in B(0, R)$ if and only if $h = r\xi$ for some $0 \leq r < R$ and some ξ in the unit sphere S^{n-1} on \mathbb{R}^n . We denote by $d\sigma(\xi)$ the surface measure on S^{n-1} . We bound the last term above by

$$\begin{aligned} & \int_0^R \int_{S^{n-1}} \left\| \frac{\mathcal{A}(\cdot + r\xi, Du) - \mathcal{A}(\cdot, Du)}{r^\alpha} \right\|_{L^p(2B)}^q d\sigma(\xi) \frac{dr}{r} \\ &= \sum_{k=0}^{\infty} \int_{r_{k+1}}^{r_k} \int_{S^{n-1}} \left\| \frac{\mathcal{A}(\cdot + r\xi, Du) - \mathcal{A}(\cdot, Du)}{r^\alpha} \right\|_{L^p(2B)}^q d\sigma(\xi) \frac{dr}{r} \\ &\leq 2^{-\alpha q} \sum_{k=0}^{\infty} \int_{r_{k+1}}^{r_k} \int_{S^{n-1}} \left\| (\tau_r \xi g_k + g_k) (1 + |Du|^2)^{\frac{1}{2}} \right\|_{L^p(2B)}^q d\sigma(\xi) \frac{dr}{r}. \end{aligned}$$

Now, using again that $Du \in L_{loc}^{p_\alpha^*}$,

$$\left\| (\tau_{r\xi} g_k + g_k) (1 + |Du|^2)^{\frac{1}{2}} \right\|_{L^p(2B)} \leq \left\| (1 + |Du|^2)^{\frac{1}{2}} \right\|_{L^{\frac{np}{n-\alpha p}}(2B)} \left\| (\tau_{r\xi} g_k + g_k) \right\|_{L^{\frac{n}{\alpha}}(2B)}$$

On the other hand, we note that for each $\xi \in S^{n-1}$ and $r_{k+1} \leq r \leq r_k$

$$\left\| (\tau_{r\xi} g_k + g_k) \right\|_{L^{\frac{n}{\alpha}}(2B)} \leq \|g_k\|_{L^{\frac{n}{\alpha}}(2B-r_k\xi)} + \|g_k\|_{L^{\frac{n}{\alpha}}(2B)} \leq 2\|g_k\|_{L^{\frac{n}{\alpha}}(\lambda B)}$$

where $\lambda = 2 + \frac{R}{r_B}$. Hence

$$\left\| \frac{\mathcal{A}(\cdot + h, Du) - \mathcal{A}(\cdot, Du)}{|h|^\alpha} \right\|_{L^q(\frac{dh}{|h|^n}; L^p(2B))} \leq C(n, \alpha, q) \left\| (1 + |Du|^2)^{\frac{1}{2}} \right\|_{L^{p_\alpha^*}(2B)} \|\{g_k\}_k\|_{\ell^q(L^{\frac{n}{\alpha}}(\lambda B))}$$

where $C(n, \alpha, q) = 2^{1-\alpha} \log 2 \sigma(S^{n-1})^{\frac{1}{q}}$. Summarizing,

$$\begin{aligned} \frac{1}{C_0} \left\| \frac{\Delta_h Du}{|h|^\alpha} \right\|_{L^q(\frac{dh}{|h|^n}; L^p(2B))} &\leq \frac{1}{r_B} \left\| \frac{\Delta_h u}{|h|^\alpha} \right\|_{L^q(\frac{dh}{|h|^n}; L^p(2B))} + \left\| \frac{\Delta_h G}{|h|^\alpha} \right\|_{L^q(\frac{dh}{|h|^n}; L^p(2B))} \\ &\quad + C(n, \alpha, q) \left\| (1 + |Du|^2)^{\frac{1}{2}} \right\|_{L^{p_\alpha^*}(2B)} \|\{g_k\}_k\|_{\ell^q(L^{\frac{n}{\alpha}}(\lambda B))} \end{aligned}$$

Lemma 7 now guarantees that $Du \in B_{p,q,loc}^\alpha$ and this concludes the proof. \square

The proofs of Theorems 2 and 4 are almost the same.

Proof of Theorem 2. Arguing again as in the proof of Theorem 3, the fact that $G = 0$ now tells us that $q \leq p_\alpha^*$ is not needed to conclude that $G_h \in L_{loc}^p$ for every single $p < \frac{n}{\alpha}$, due to Theorem 5. As a consequence, (5.3) holds for every $p < \min\{p_0, \frac{n}{\alpha}\}$. The rest of the proof follows in the same way. \square

Proof of Theorem 4. Arguing again as in the Proof of Theorem 3, the new equation \mathcal{A}_h is now linear with VMO coefficients, due to the linearity of $\mathcal{A}(x, \xi)$ as a function of ξ . Also, from $\max\{1, \frac{nq}{n+\alpha q}\} < p < \frac{n}{\alpha}$ we have $q \leq p_\alpha^* < \infty$ and so $G \in L_{loc}^{p_\alpha^*}$ implies $Du \in L_{loc}^{p_\alpha^*}$ by the results at [14]. Hence, G_h has an L_{loc}^p majorant, and thus $Dv_h \in L_{loc}^p$ again by [14], since $p > 1$. In particular, the restriction $p < \min\{p_0, \frac{n}{\alpha}\}$ can be replaced by $p < \frac{n}{\alpha}$, and the restriction $p > p'_0$ can be replaced by $p > \max\{1, \frac{nq}{n+\alpha q}\}$. The rest of the proof follows similarly. \square

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A. L. Baisón, A. Clop, J. Orobitg
Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193-Bellaterra (CATALONIA)

R. Giova
Dipartimento di Studi Economici e Giuridici
Università degli Studi di Napoli "Parthenope"
Palazzo Pacanowsky- Via Generale Parisi, 13
80123 Napoli (Italy)

A. Passarelli di Napoli
Dipartimento di Matematica e Appl. "R.Caccioppoli"
Università degli Studi di Napoli "Federico II"
Via Cintia, 80126 Napoli (Italy)